Discrete breathers for a discrete nonlinear Schrödinger ring coupled to a central site

Peter Jason* and Magnus Johansson†

Department of Physics, Chemistry and Biology (IFM), Linköping University, SE-581 83 Linköping, Sweden
(Dated: January 12, 2016)

Abstract

We examine the existence and properties of certain discrete breathers, for a discrete nonlinear Schrödinger model where all but one sites are placed in a ring and coupled to the additional, central site. The discrete breathers we focus on are stationary solutions mainly localized on one or a few of the ring-sites, and possibly also the central site.

By numerical methods, we trace out and study the continuous families the discrete breathers belong to. Our main result is the discovery of a "split-bifurcation" at a critical value of the coupling between neighboring ring sites. Below this critical value, families form closed loops in a certain parameter space, implying that discrete breathers with and without central-site occupation belong to the same family. Above the split-bifurcation the families split up into several separate ones, which bifurcate with solutions with constant ring-amplitudes. For symmetry reasons, the families have different properties below the split bifurcation for even and odd number of sites. It is also determined under which conditions the discrete breathers are linearly stable.

The dynamics of some simpler initial conditions which approximate the discrete breathers are also studied, and the parameter regimes where the dynamics remain localized close to the initially excited ring-site are related to the linear stability of the exact discrete breathers.

PACS numbers:

*Electronic address: petej@ifm.liu.se
†Electronic address: mjn@ifm.liu.se; URL: https://people.ifm.liu.se/majoh
I. INTRODUCTION

The existence of localized solutions is a much studied property of nonlinear models with translational symmetry. In nonlinear lattice models this may occur in the form of discrete breathers (DBs), which are not only spatially localized but also oscillating (‘breathing’) in time [1, 2]. Even though DBs had been studied earlier [3], it was with the seminal paper by MacKay and Aubry [4] that their generic nature was revealed. They showed that by starting with localized solutions in the so-called anti-continuous limit, where all the inter-site coupling is turned off and localized solutions therefore can be obtained trivially, continuous families of DBs can be followed for increasing coupling strength under quite general conditions. Since then, additional proofs of the existence of DBs have been put forward [2]. A crucial point in the proof of Ref. [4] is that because of the discreteness, the linear extended phonons exist in limited bands, which makes it possible for the DBs to avoid resonances with them. This is also the reason why breathers are not generic in continuous nonlinear models, where the phonon spectrum is unbounded and resonances generally are impossible to avoid (exceptions exist, such as for the integrable sine-Gordon model [5] and continuous nonlinear Schrödinger equation [6]).

One of the most widely studied nonlinear lattice models is the discrete nonlinear Schrödinger (DNLS) equation [7]. It has in recent years gained much attention for describing Bose-Einstein condensates in optical lattices [8] and optical wave-guide arrays [9, 10]. We will in this paper consider a DNLS model with the geometry of a ring coupled to a single central site, which may serve as a model for a type of optical multicore fiber [11, 12]. Multicore fibers have been proposed as a promising candidate for scaling up the transmission capacity for optical communications [13, 14], and experimental realizations of the type the model used in this paper would apply for include the seven-core fiber [11, 14] (i.e. six ring-sites). We will therefore use this particular configuration frequently throughout the paper.

The model under consideration has also been studied in Refs. [15, 16], focusing on stationary solutions where all the ring-sites have the same amplitude. These are referred to as ‘dimerlike solutions’, because the model effectively reduces to a dimer. They found that there exist stable solutions which have most of the amplitude located on the central site, and consequently can be considered as localized solutions. This work has been extended to a model which also included gain and loss [17]. Recently there have also been studies
of discrete vortices for the DNLS model [18]. Ref. [19] uses the DNLS model to describe a strong control field, to which a weaker signal field is coupled, studying initial conditions localized on one ring-site and the central site. Refs. [20, 21] studied a model related to the DNLS, the discrete Ginzburg-Landau equation, and how localization can arise through instabilities for solutions where all ring-sites have the same amplitude.

Andersen and Kenkre [22] found analytical solutions for some special types of initial conditions to a DNLS model where all sites are coupled to each other with equal strength. Concerning the model in this paper, these results would be applicable to a configuration with four equivalent sites.

We will instead examine the existence and properties of stationary DBs which are mainly localized on a few sites of the ring, and possibly also the central site. We are however dealing with finite systems, and should therefore be a bit cautious when using terms such as DBs and localized. More specifically, we study the continuous families which can be smoothly followed from solutions that in the anti-continuous limit only have non-zero amplitudes on a few of the ring-sites, and possibly also the central site. With DBs we therefore mean solutions with a non-constant amplitude profile which belong to these families. The localization of these solutions will range from having essentially all norm on one or few sites, to being almost entirely spread out. But since there is a smooth transition between them, we say that they all belong to the DB-family.

A different viewpoint of the objectives would be to say that we will study how the DBs of a one-dimensional, periodic lattice are affected when all sites get coupled to an additional central site.

The outline of the paper is as follows. In Sec. II the model is introduced, and in Sec. III we discuss the method used to numerically trace out the continuous families of DBs. In Sec. IV we discuss DBs with the ring and central site decoupled from each other, beginning by reviewing some earlier results for the simple ring geometry. In Sec. V the coupling between the ring and central site is turned on, and in Sec. VA we trace out the continuous subfamilies that the DBs belong to as a function of this coupling parameter. We proceed by examining the linear stability of the solutions in Sec. VB, studying the dynamics of some simpler initial conditions approximating the exact DBs in Sec. VC, and analyzing a bifurcation which splits up the continuous DB-subfamilies in Sec. VD. Sec. VI finishes the paper with a summary and conclusions.
II. MODEL

We will consider a DNLS model with the geometry of a $N_R$-site ring, all ring-sites equivalent, coupled to an additional central site

\[
\frac{i}{\hbar} \frac{d\psi_n}{dt} + \beta (\psi_{n+1} + \psi_{n-1}) + |\psi_n|^2 \psi_n + \delta \psi_C = 0, \tag{1a}
\]

\[
\frac{i}{\hbar} \frac{d\psi_C}{dt} + \delta \sum_{n=1}^{N_R} \psi_n + \gamma |\psi_C|^2 \psi_C = 0, \tag{1b}
\]

where $\psi_n$, $n = 1, \ldots, N_R$, denote the generally complex amplitudes of the ring-sites, $\psi_C$ the corresponding quantity for the central site, and because of the periodicity of the ring $\psi_{N_R+1} = \psi_1$ and $\psi_0 = \psi_{N_R}$. Further, $t$ is the propagation parameter which represents propagation length in the context of optical wave guides and time in the context of Bose-Einstein condensates, $\beta$ is the linear coupling parameter between neighboring ring-sites, $\delta$ the linear coupling parameter between one ring-site and the central site, and $\gamma$ the nonlinear on-site parameter for the central site. The parameters are all given relative to the nonlinear on-site parameter for the ring-sites, which is fixed to unity.

The DNLS model supports stationary solutions of the type $\{\psi_n(t), \psi_C(t)\} = \{\phi_n, \phi_C\} e^{i\omega t}$, where $\{\phi_n, \phi_C\}$ are $t$-independent amplitudes, which themselves are solutions to the algebraic equations

\[
-\omega \phi_n + \beta (\phi_{n+1} + \phi_{n-1}) + |\phi_n|^2 \phi_n + \delta \phi_C = 0, \tag{2a}
\]

\[
-\omega \phi_C + \delta \sum_{n=1}^{N_R} \phi_n + \gamma |\phi_C|^2 \phi_C = 0. \tag{2b}
\]

This corresponds to moving into a co-rotating frame, in which the DB becomes truly stationary. These solutions are in the context of optics describing light propagating with the same wavelength in all wave-guides and with unchanging amplitudes, and they will be the focus of this paper. The model has two conserved quantities: the Hamiltonian

\[
H = -\left( \sum_{n=1}^{N_R} \left[ \beta (\psi_{n+1}^* \psi_n + \psi_n^* \psi_{n+1}) + 0.5 |\psi_n|^4 + \delta (\psi_C^* \psi_n + \psi_n^* \psi_C) \right] + 0.5 \gamma |\psi_C|^4 \right), \tag{3}
\]

with the associated canonical positions $\{\psi_n, \psi_C\}$ and momenta $\{i\psi_n^*, i\psi_C^*\}$, and the norm

\[
\mathcal{N} = \sum_{n=1}^{N_R,C} |\psi_n|^2, \tag{4}
\]

where the conservation of the latter is related to the U(1) symmetry of the model.
III. METHOD

The method we will employ to numerically trace out continuous families of DBs is quite standard [23]. The basic idea is to start in the anti-continuous limit, where all inter-site couplings are zero and localized solutions are trivially obtained by exciting some sites while keeping the others equal to zero. Such a solution can then be used as the initial guess in a Newton-Raphson algorithm to numerically find a solution to Eq. (2), but with the coupling constants turned up slightly. By repeating this procedure, and using the solution calculated in the previous step as the initial guess, it is possible to follow continuous families of DBs as functions of the coupling constants.

Note though that unless the U(1) symmetry of the DNLS model is appropriately handled, the functional matrix (which is inverted in the Newton-Raphson algorithm) will be singular. This is resolved by explicitly breaking the symmetry by specifying the overall phase of \( \{ \phi_n, \phi_C \} \), which for convenience is taken to be real. This is however not viable in all situations, for instance when working with families of quasiperiodic solutions or vortices, where one instead might use a pseudoinverse [24].

If we were to solve only Eq. (2) with the Newton-Raphson algorithm, the parameters (\( \omega, \beta, \delta \) and \( \gamma \)) would all be specified beforehand, while the \( N_R + 1 \) real amplitudes \( \{ \phi_n, \phi_C \} \) would be treated as variables, and thus determined by the algorithm. We will however add an extra constraint, which states that the norm \( N \) of the DBs should be equal to unity. This is in practice done by also solving \( \sum_{n=1}^{N_R+C} |\phi_n|^2 - 1 = 0 \) together with Eq. (2), which implies that one of the parameters now must be treated as a variable. Note though that this is not a ‘physical’ restriction, since the model is invariant under the variable substitution \( \{ \phi_n', \phi_C' \} = \{ \sqrt{\alpha} \phi_n, \sqrt{\alpha} \phi_C \} \) together with a redefinition of the parameters \( \{ \omega', \beta', \delta', \gamma' \} = \{ \alpha \omega, \alpha \beta, \alpha \delta, \gamma \} \). Other parameter substitutions are allowed, as long as the ratio between a new linear and nonlinear parameter is \( \alpha \) times the ratio between the corresponding old ones. The substitution given keeps the nonlinear parameter for the ring-sites equal to unity.

Since there are two inter-site coupling constants in the model, we will employ a two-step scheme to trace out the DB-families. We begin by only increasing the ring-coupling \( \beta \), while using \( \omega \) as a variable, which effectively gives us DBs for a one-dimensional periodic model (the decoupled central site is however not necessarily empty). We will only consider positive \( \beta \) in this paper, but note that changing \( \beta \rightarrow -\beta \) together with a staggering transformation
of the ring $\phi_n \rightarrow (-1)^n \phi_n$ leaves the model invariant, and all our results for an even number of ring-sites can easily be transferred to the corresponding model with negative $\beta$. For an odd number of ring-sites the staggering transformation is not viable, and negative values of $\beta$ must therefore be treated separately (which is not done in this paper).

The second step will be to pick out DBs for certain values of $\beta$ and trace out the DB-subfamilies as functions of $\delta$. As mentioned above, it is necessary to treat one of the parameters as a variable since the norm is fixed, and the natural choice is arguably $\omega$. It was however found during the second step of our strategy that the DBs do not necessarily exist in subfamilies which can be followed for monotonically increasing $\delta$ [see e.g. Fig. 1]. We will therefore alternate the roles between $\omega$ and $\delta$, and use one of them as the variable in the Newton-Raphson algorithm, while increasing or decreasing the other in small increments.

IV. DISCRETE BREATHERS FOR UNCOUPLED RING AND CENTRAL SITE

The central site will obviously remain zero if we start with a solution without any central amplitude when increasing $\beta$ with $\delta = 0$. One will thus obtain exactly the same DB-subfamily as one would for a one-dimensional, periodic model. The existence and properties of these solutions have been studied extensively, and we will recall some of the results which are most relevant for our work [25, 26].

The family of on-site symmetric DBs that starts with $\{\phi_n, \phi_C\} = \{\delta_{n,k}, 0\}$ in the anti-continuous limit will for $N_R \geq 6$ bifurcate with the uniform-ring solution $\{\phi_n, \phi_C\} = \{\sqrt{1/N_R}, 0\}$ at [25, 26] 

$$\beta = \beta_{ring}(N_R) = \frac{1}{2N_R \sin^2(\pi/N_R)}. \quad (5)$$

This is related to the onset of a modulational instability for the uniform-ring solution.

For $N_R = 3, 4, 5$ the family instead bifurcates with another single-peak solution at a $\beta = \beta^*(N_R) > \beta_{ring}(N_R)$. The DBs in these families are always linearly stable and almost always the ground states when they exist. The exception is once again for $N_R = 3, 4, 5$ where the uniform-ring solution becomes the ground state in a region close to the largest $\beta$ for which the DBs exist. The uniform-ring solution is also the ground-state for all lattice sizes when $\beta$ is so large that the DBs do not exist [25, 26]. We will refer to the on-site symmetric DBs for $\delta = 0$, with zero central amplitude, that can be followed from a single excited ring-site in the anti-continuous limit as an on-site ring-DB.
The family of DBs for $\delta = 0$ that starts with $\{\phi_n, \phi_C\} = \{\delta_{n,k}/\sqrt{2} + \delta_{n,k+1}/\sqrt{2}, 0\}$ in the anti-continuous limit (referred to as inter-site ring-DBs) will also bifurcate with the uniform-ring solution at $\beta_{\text{ring}}(N_R)$, but also for $N_R < 6$. The single-peak solution that the on-site symmetric DBs bifurcate with for $N_R = 3, 4, 5$ is actually ‘created’ in the bifurcation between the inter-site DB and uniform-ring solution [26].

Given a pure ring-DB with frequency $\omega$, we may generate a stationary unit-norm DB with nonzero central amplitude $\phi_C$ (still at $\delta = 0$) by a simultaneous rescaling of the amplitudes $\{\phi_n, \phi_C\}$, ring-coupling $\beta$, and frequency $\omega$. Requiring, for a given central-site nonlinearity $\gamma$, Eqs. (2a) and (2b) to be simultaneously fulfilled for the same $\omega$ yields the transformation:

$$
\{\phi_n, \phi_C = 0\} \rightarrow \{\phi'_n, \phi'_C\} = \{\phi_n/\sqrt{1 + \omega/\gamma}, 1/\sqrt{1 + \gamma/\omega}\},
$$

$$
\beta \rightarrow \beta' = \beta/(1 + \omega/\gamma),
$$

$$
\omega \rightarrow \omega' = \omega/(1 + \omega/\gamma),
$$

with primed variables for the solution with nonzero central amplitude. Note that $\gamma$ must be specified in the transformation, since $\omega$ on the central site depends on this parameter, according to Eq. (2b). By utilizing this transformation, one may obtain from the corresponding families without central occupation, whole families of stationary DBs with constant unity norm and nonzero central amplitude. These are parametrized by $\omega/\gamma$, with $\phi_C \rightarrow 0$ when $\omega/\gamma \rightarrow 0$, and $\phi_C \rightarrow 1$ when $\omega/\gamma \rightarrow \infty$. Eq. (5) then implies that the ($\delta = 0$) families with nonzero central amplitudes, that are obtained from the on-site (for $N_R > 6$) and inter-site ring-DBs using Eq. (6), will bifurcate with dimerlike solutions $\{\phi_n = \text{const}, \phi_C\}$ at

$$
\beta_{\text{dimer}}(N_R, \gamma) = \frac{\gamma}{2((\gamma N_R + 1) \sin^2(\pi/N_R) + 1)},
$$

and at the bifurcation point have the frequency

$$
\omega_{\text{dimer}}(N_R, \gamma) = \frac{\gamma \sin^2(\pi/N_R) + 1}{(\gamma N_R + 1) \sin^2(\pi/N_R) + 1}.
$$

Note that $\beta_{\text{ring}}(N_R) = \lim_{\gamma \to \infty} \beta_{\text{dimer}}(N_R, \gamma)$. 

7
FIG. 1: (Color online) Curves in the $\delta - \omega$ parameter space of DB-subfamilies connected to the on-site ring-DBs, for $N_R = 6$ and $\gamma = 1$. (a) $\beta < \beta^{\text{split}}(N_R, \gamma)$. Going from the outer orange (gray) to the inner black curve, $\beta$ range from 0.0264 to 0.1452 in steps of 0.0297. The outer orange (gray) curve does not form a closed loop, but converges towards a solution with $\sum_{n=1}^{N_R} \phi_n = 0$. (b) $\beta > \beta^{\text{split}}(N_R, \gamma)$. The curves bifurcate with dimerlike solutions, marked with black dots. $\beta$ range from 0.1485 (outer yellow or light gray curve) to 0.3267 (inner black curve), with an increase of 0.0099 between nearby curves.

V. DISCRETE BREATHERS FOR COUPLED RING AND CENTRAL SITE

A. Discrete breather families

We will in this section execute the second step of the scheme described in Sec. III, and trace out DB-subfamilies with fixed $\beta$ as functions of $\delta$. The properties of the DB-subfamilies that can be followed from the on-site ring-DB will depend on the value of $\beta$, $N_R$ and $\gamma$. There are e.g. certain differences between $N_R$ even or odd, and if $N_R \geq 6$, and these cases are treated separately below.
1. Even \( N_R \)

The DB-families connected to the on-site ring-DBs exhibit what we call a \textit{split-bifurcation} for a certain value \( \beta = \beta_{\text{split}}(N_R, \gamma) \). The DB-subfamilies bifurcate with dimerlike solutions for \( \beta > \beta_{\text{split}}(N_R, \gamma) \), while they can be followed along closed loops in the \( \delta - \omega \) parameter space for \( \beta < \beta_{\text{split}}(N_R, \gamma) \). Fig. 1 illustrates how the subfamilies of the on-site ring-DB change with \( \beta \) for the particular case of \( N_R = 6 \) and \( \gamma = 1 \), with Figs. 1(a) and 1(b) being below and above the split-bifurcation, respectively. The on-site ring-DBs are located at the very top of the plotted curves. When \( \delta = 0 \), the on-site ring-DB bifurcates with the uniform-ring solution at \( \beta_{\text{ring}} = 1/3 \) [Eq. (5)], while the corresponding DB with central occupation bifurcates with the dimerlike solution at \( \beta_{\text{dimer}} = 2/11 = 0.1818... \) [Eq. (7)], for the values \( N_R = 6 \) and \( \gamma = 1 \) of Fig. 1. For \( N_R \geq 6 \) the subfamilies bifurcate with dimerlike solutions for all \( \beta_{\text{split}}(N_R, \gamma) < \beta < \beta_{\text{ring}}(N_R) \). The closed curves have the same general shape in the \( \delta - \omega \) plane for different even \( N_R \) below the split-bifurcation. These results have been numerically tested up to \( N_R = 50 \). Note also that the maximum value of \( \delta \) for the subfamilies decreases with increasing \( \beta \), approaching \( \delta = 0 \) as \( \beta \to \beta_{\text{ring}}(N_R) \). Another bifurcation, a \textit{close-bifurcation}, occurs for a \( \beta < \beta_{\text{split}}(N_R, \gamma) \) (cf. outer orange (gray) curve in Fig. 1(a)), and will be discussed later in the section.

Fig. 2(b) shows the solutions that belong to the subfamily in Fig. 1(a) with \( \beta = 0.1155 \), replotted in Fig. 2(a). Note that the closed loops do not display crossings at points (\( \text{(ii)} \)) in Figs. 2(a) and 2(c), since although the corresponding solutions are equivalent, they have different phases between central and ring-sites, and are localized around different ring-sites (same is true for \( \text{(ii)} \) and \( \text{(iv)} \) in Fig. 3(a)). The markings indicate how the curves should be followed. The general features of how the solutions change along different parts of a closed curve below the split-bifurcation are the same for different even \( N_R, \beta \) and \( \gamma \) (\( \beta_{\text{split}}(N_R, \gamma) \) do depend on \( N_R \) and \( \gamma \), see Fig. 8 and Sec. V D). Extensive calculations have been performed to verify this result (and the corresponding one for odd \( N_R \)) for \( N_R \) between 3 and 50 sites, \( \gamma \) between zero and \( 10^6 \), and \( \beta \) between zero and \( \beta_{\text{ring}} \). For example, the main effect when going from the on-site ring-DB at the top of the curve down to the solution marked with (\( \text{(ii)} \)) is to increase the amplitude on the central site, while (at least initially) spreading out the solution over the ring. Note that maximal spread occurs for an intermediate point close to (\( \text{(ii)} \)), which is where the subfamily splits up at the split-bifurcation (see Sec. V D). Above
FIG. 2: (Color online) (a) Subfamily of the on-site ring-DBs in $\delta - \omega$ diagram for $N_R = 6$, $\beta = 0.1155$, and $\gamma = 1$. The black parts of the curve indicate where the DBs are linear stable, while the yellow (light gray) and orange (dark gray) indicate linear instability. For yellow (light gray) all unstable eigenvalues are real, while for orange (dark gray) some of them are complex. (b) How $\{\phi_n, \phi_C\}$ change with $\omega$ along the curve in (a), when starting at the on-site ring-DB localized at site 4, located at the top of (a), and following the path indicated by the markings (included as vertical lines). The $\omega$-axis is folded around the solid, vertical line of solution (iii). (c)-(d) Similar to (a) and (b), and for the same parameter values, but for the subfamily of the inter-site ring-DB, which is located at the top of the thick curve in (c). The subfamily in (a) is included in (c) as a thin, blue (gray) line.

the split-bifurcation, the DBs change in qualitatively the same way as the DBs do along the corresponding part of a connected curve below the split-bifurcation. For the subfamily plotted in Fig. 1(b), the DBs thus spread out over ring while the central amplitude increases, when followed from the on-site ring-DB towards the dimerlike solution. That the subfamilies
tend to bifurcate with the dimerlike solution for larger $\beta$ is quite intuitive, since larger $\beta$ corresponds to on-site ring-DBs which are more spread out over the ring, and consequently 'closer' to a dimerlike solution.

Figs. 2(c)-(d) show plots which correspond to Figs. 2(a)-(b) for the inter-site ring-DB, and for the same parameter values. The subfamily in Fig. 2(a) has been included also in Fig. 2(c) as a thin blue (gray) line as a reference. The subfamily of the inter-site ring-DB also exhibits a split-bifurcation, at the same $\beta = \beta^{\text{split}}(N_R, \gamma)$ as the subfamily of on-site ring-DB. The two subfamilies bifurcate with the same dimerlike solution, at the same $\delta$, above the split-bifurcation. The mechanism of the split-bifurcation is analyzed further in Sec. V D.

The on-site (inter-site) symmetry of the on-site (inter-site) ring-DB is preserved for all solutions in the associated subfamily (applies also for odd $N_R$). Some solutions, e.g. the ones located at the bottom of Fig. 2(a) and the ones slightly to the left of solution (iii) in Fig. 3(b), have two peaks located on opposite sides of the ring. These can either be considered as multi-DBs (i.e. several separated excited sites in the anti-continuous limit), or as DBs that are localized diagonally across the ring, since also the central site is excited.

The mirror symmetry around the $\omega$-axis in Fig. 1(a), Fig. 2(a) and Fig. 2(c) is due to that changing the sign on both $\phi_C$ and $\delta$ leaves the model invariant, and the ‘mirrored’ solution will thus have the opposite relative sign between ring-sites and central site. This property holds also for odd $N_R$. There is a mirror image for Fig. 1(b) (omitted for clarity).

The solution for $N_R = 6$ at (iii) in Fig. 2(a)-(b) can be obtained analytically by applying transformation (6) to a solution called $[\uparrow \ldots \downarrow \ldots]$ in Ref. [27] and ‘dimerlike’ in Ref. [28] (distinct from the solution we call dimerlike). Also solution (iii) in Fig. 2(c)-(d) can be obtained analytically with transformation (6), this time applied to the ‘single depleted well’ solution of Ref. [28], resulting in $\phi_{3,4} = -\phi_{1,6} = \sqrt{(\gamma - \beta)/(1 + 4\gamma)}$, $\phi_{2,5} = 0$, and $\phi_C = \sqrt{(1 + 4\beta)/(1 + 4\gamma)}$. Ref. [28] actually treats only the trimer with negative $\beta$, but the solutions can be transferred to $N_R = 6$ and positive $\beta$, as shown in Ref. [29].

As stated above, a close-bifurcation occurs for the on-site ring-DBs at $\beta = \beta^{\text{close}}(N_R, \gamma) < \beta^{\text{split}}(N_R, \gamma)$. For $\beta < \beta^{\text{close}}(N_R, \gamma)$, the subfamilies of the on-site ring-DB do not form closed loops in the $\delta - \omega$ plane (e.g. outer orange (gray) curve in Fig. 1(a)), which they do for $\beta^{\text{close}}(N_R, \gamma) < \beta < \beta^{\text{split}}(N_R, \gamma)$. Below the close-bifurcation, the subfamilies converge towards solutions where both $\sum_{n=1}^{N_R} \phi_n$ and $\phi_C$ approach zero for large $|\delta|$, while $\delta \phi_C$ ap-
approaches a constant nonzero value. These solutions correspond to one-peak DBs with the amplitudes shifted down (assuming a positive peak). They can be followed from solutions in limits with $\beta = 0$ and $\delta \phi_C \neq 0$, where one ring-site (the peak-site) has a large positive amplitude and the other ring-sites have small and identical negative amplitudes. This corresponds essentially to a ring with $\delta \phi_C$ acting as an external driving. The curves approach horizontal lines in the $\delta - \omega$ plane for large $|\delta|$ (the outer orange (gray) curve in Fig. 1(a) approaches $\omega \approx 0.66$). Note also that the closed subfamilies break up at different parts of the curve when going below and above the close- and split-bifurcation, respectively. The on-site ring-DB and the corresponding DB with central amplitude (solution (ii) in Fig. 2(a)-(b)) belong to the same subfamily below the close-bifurcation, which they do not above the split-bifurcation.

For $\beta < \beta^O_{\text{OS, close}}(N_R, \gamma)$, the subfamily of solution (iii) in Fig. 2(a) bifurcates with a solution with $\sum_{r=1}^{N_R} \phi_n = 0$ and $\phi_C = 0$, for which the central site and ring are decoupled, and $\omega$ does not depend on $\delta$. The solution consists of two one-site peaks in anti-phase located on either side of the ring, and for the ring-sites $\phi_n = -\phi_{n+N_R/2}$. For the case $N_R = 6$, this is the $[\uparrow \ldots \downarrow \ldots]/$dimerlike solution of Refs. [27, 28].

A close-bifurcation occurs also for the subfamily of the inter-site ring-DB, for $\beta = \beta^{I\text{S, close}}_{\text{close}}(N_R, \gamma) < \beta^O_{\text{close}}(N_R, \gamma)$, however only for $N_R \geq 7$ (also for odd $N_R$, see Fig. 8(b)). The subfamily converges to a solution which is similar to the corresponding one for the on-site ring-DB, but with the peak consisting of two equally occupied neighboring sites. The subfamily also breaks up at an analogous point on the curve, close to the turn between solutions (ii) and (iii) in Fig. 2(c).

Between the close- and split-bifurcation, following a closed loop (one revolution) in the $\delta - \omega$ plane results in a translation of the ring-amplitudes by $N_R/2$ (i.e. the ring rotates half a turn), and a global phase-shift of $\pi$. This could for instance be used to transfer the peak of an on-site ring-DB to the other side of the lattice, which may be interesting for applications in e.g. optical communication. Note though that the subfamilies are linearly unstable along large parts of the curves (see Sec. V B).
FIG. 3: (Color online) Similar to Fig. 2(a)-(b) but for \( N_R = 7, \beta = 0.1147, \) and \( \gamma = 1. \) (a) Inset: Subfamily of the inter-site ring-DB for \( N_R = 7, \gamma = 1, \) and \( \beta^{IS} (N_R, \gamma) < \beta = 0.0555 < \beta^{OS} (N_R, \gamma) \) in \( \delta - \omega \) plane, with the inter-site ring-DB located at the top of the curve. Linear stability not indicated. (b) Only \( \left\{ \phi_n, \phi_C \right\} \) for half of the curve in the main figure of (a) is plotted, from the on-site DB (localized at site 4) located at the top of (a), to the inter-site ring-DB (v) (localized at sites 1 and 7). The solutions for the other ‘mirrored’ part of the curve are obtained by changing the sign of the central site, with no translation of the ring-sites. The \( \omega \)-axis is folded around solution (iii).

2. Odd \( N_R \)

There are many similarities between odd and even \( N_R, \) e.g. the occurrence of a split-bifurcation at some \( \beta^{split} (N_R, \gamma) \), and of close-bifurcations at \( \beta^{IS}_{close} (N_R, \gamma) < \beta < \beta^{OS}_{close} (N_R, \gamma) < \beta^{split} (N_R, \gamma) \). There are however some other types of DBs that belong to the closed subfamilies between the close- and split-bifurcations, and the associated curves in the \( \delta - \omega \) plane therefore have different shapes. For instance, the on-site and inter-site ring-DB belong to the same subfamily for \( \beta^{OS}_{close} (N_R, \gamma) < \beta < \beta^{split} (N_R, \gamma) \). This difference between odd and even \( N_R \) can be attributed to the preservation of on-site symmetry: it is possible when \( N_R \) is odd for solutions to be simultaneously on-site symmetric around one site, and inter-site symmetric around the point located in-between the two sites on the opposite side of the ring. Only dimerlike solutions are simultaneously on-site and inter-site symmetric for even \( N_R \). As for even \( N_R \), the on-site and inter-site ring-DB bifurcate with the same dimerlike
solution, at the same $\delta$, above the split-bifurcation.

Fig. 3(a) plots a closed subfamily between the on-site close- and split-bifurcation for $N_R = 7$, $\beta = 0.1147$ and $\gamma = 1$ in the $\delta - \omega$ plane, with the associated solutions shown in Fig. 3(b). The on-site ring-DB is located at the top of the curve in Fig. 3(a), and the inter-site ring-DB is marked with $(v)$ (note the location of their peaks in Fig. 3(b)). Fig. 3(b) follows the curve in Fig. 3(a) by first going down in $\omega$ from the on-site ring-DB to solution $(iii)$, and then up in $\omega$ to the inter-site ring-DB $(v)$, with the marks in Fig. 3(a) indicating the path followed.

For $\beta_{\text{close}}^{IS}(N_R, \gamma) < \beta < \beta_{\text{close}}^{OS}(N_R, \gamma)$, the inter-site ring-DB belong to a subfamily which forms a (different) closed loop in the $\delta - \omega$ plane (see inset of Fig. 3(a)). Below the associated close-bifurcation, the subfamilies of the on-site and inter-site ring-DBs converge towards the same type of solutions as when $N_R$ is even, i.e. DBs with the ring-site amplitudes shifted down and peaks consisting of one and two neighboring sites, respectively. The closed curves also break up along analogous parts of the curve, being close to the turn between solutions $(ii)$ and $(iii)$ in Fig. 3(a) for the on-site ring-DB, and close to solution $(iii)$ in Fig. 3(a) for the inter-site ring-DB.

Similar to the case of even $N_R$, the general shape of the closed curves between the close- and split-bifurcations, and how the DBs qualitatively change along different parts, will not depend on the values of $\beta$, $\gamma$ or (odd) $N_R$. How $\beta_{\text{split}}^{OS}(N_R, \gamma)$, $\beta_{\text{close}}^{OS}(N_R, \gamma)$ and $\beta_{\text{close}}^{IS}(N_R, \gamma)$ depend on $N_R$ and $\gamma$ is treated in Sec. V D (see Fig. 8). In contrast to even $N_R$, following a closed curve adiabatically a full revolution will not result in a translation of the amplitudes, but only a phase shift of $\pi$. Any translation, apart from over the whole ring, would violate the on-site symmetry for odd $N_R$.

3. Close-bifurcation for $N_R < 6$

Another close-bifurcation occurs for $N_R < 6$, at a $\beta > \beta_{\text{split}}^{IS}(N_R, \gamma)$, above which the subfamilies of the on-site ring-DB have closed up again and form closed loops. This close-bifurcation occurs at $\beta_{\text{ring}}(N_R)$ [Eq. (5)] when the inter-site ring-DB bifurcates with the uniform ring solution (for $\delta = 0$). Remember that the on-site ring-DB bifurcates with another one-peak solution, instead of the uniform ring solution, for $N_R < 6$ [26]. The close-bifurcation is illustrated for $N_R = 5$ and $\gamma = 1$ in Fig. 4. The subfamilies have a circular
shape in the $\delta - \omega$ plane above the close-bifurcation, with two different $\delta = 0$ solutions for each subfamily: the on-site ring-DB, located at the top of the curves in Fig. 4(a), and the one-peak solution it eventually bifurcates with (for $\delta = 0$). The subfamilies circular curve in the $\delta - \omega$ plane will shrink to a single point as the latter bifurcation is approached.

B. Linear Stability

The linear stability of the DBs is also shown in Figs. 2(a), 2(c) and 3(a), with black meaning linearly stable while yellow (light gray) and orange (dark gray) both indicate linear instability. The linear stability is determined by the eigenvalues of the matrix obtained by linearizing Eq. (1) (see e.g. Ref [30] for further details), with the convention that linearly unstable eigenmodes have eigenvalues with positive, nonzero real part. The matrix in question is infinitesimally symplectic [31], and if $\lambda$ is an eigenvalue, then so are also $-\lambda$, $\lambda^*$ and $-\lambda^*$. Purely real or imaginary eigenvalues will therefore occur in pairs, while other complex eigenvalues occur in quadruplets.
FIG. 5: (Color online) (a)-(c) Evolution of $|\psi_n(t)|^2$ for randomly perturbed unstable stationary solutions. The thick orange (dark gray) curve is the central site, the thick black curve is the initially most occupied ring-site, while the thin lines represent all other ring-sites. The parameters for the unstable stationary solutions are the same as in Fig. 2 but with (a) $\delta = 0.1321$, $\omega = 0.7104$, and (b)-(c) $\delta = 0.0944$, $\omega = 0.6104$ (i.e. same parameters for (b) and (c) but different perturbations). (d) Fraction of trajectories originating from randomly perturbed unstable DBs where localization occurs on central site (orange or gray) or the initially most occupied site (black). The DBs are located between (i) and (ii) in Fig. 2(a)-(b). One thousand simulations were performed for each considered $\omega$.

The yellow (light gray) segments of the curves in Figs. 2(a), 2(c) and 3(a) mark solutions with only real unstable eigenvalues, while for orange (dark gray) at least some of them are complex. When starting at the on-site ring-DB, and following the subfamilies for increasing or decreasing $\delta$, the solutions remain linearly stable until the first bend of the curve. We have found that this is generic for the DB-subfamilies [cf. Fig. 1], also for other $\gamma$ and $N_R$ (numerically tested for $3 \leq N_R \leq 50$, and $0 \leq \gamma \leq 10^4$).

The subfamilies are also linearly stable in one direction for small $\delta$ close to the on-site symmetric DBs with central occupation (solutions marked with (ii) in Figs. 2(a), 2(b) and 3). The criterion for stability is similar to that of the twisted localized modes [32], so that solutions are stable (unstable) for positive $\delta$ if the central site and the mostly occupied ring-site are in anti-phase (in-phase), and vice versa for negative $\delta$.

Dynamical simulations show that there is a tendency for localization to occur mainly on a single site when a small random perturbation is added to the unstable solutions located between solutions (i) and (ii) in Figs. 2(a), and 3(a). They also show that for solutions with
large $\omega$ (still between (i) and (ii)) the main localization remains on the initially most excited ring-site, illustrated in Figs. 5(a) for $\omega = 0.7104$, $\delta = 0.1321$, $\gamma = 1$ and $N_R = 6$. This is quite intuitive since the (unperturbed) solutions are close to the upper stable branch in Figs. 2(a) and 3(a). For solutions further down on these parts of the curves, having a larger central amplitude, the main localization may instead occur on the central site. A perturbation may then result in localization on either the central site or on the mostly excited ring-site, with the form of the perturbation determining which, as illustrated in Figs. 5(b) and 5(c), for $\omega = 0.6104$, $\delta = 0.0944$. For Fig. 5(d), dynamical simulations were performed with small random perturbations added to DBs located between solutions (i) and (ii) in Figs. 2(a), the figure showing the fraction of the simulations where localization have occurred on the central and initially most occupied ring-site, respectively. This is determined by running the simulations until $t = 200$, and if one site is the mostly occupied site for all time steps between $t = 100$ and $t = 200$, we say that localization occurs on this site. There is a possibility that the perturbed DBs do not get localized on a single site for a longer period of time, but instead ‘jumps’ between different sites. This happened quite rarely for Fig. 5(d), typically for less than 3% of the simulations, apart from close to (ii) ($0.50 \lesssim \omega \lesssim 0.52$) where it occurred for approximately 15% or less of the simulations. The size of the applied perturbations are $\sum_{n=1}^{N_R} C_n |\psi_n(0) - \phi_n|^2 \approx 10^{-5}$, where $\{\phi_n\}$ is the unstable DB and $\{\psi_n(0)\}$ the result when randomly perturbing it. One thousand simulations were performed for each unperturbed DB considered. We have used Adams-Bashforth-Moulton solver of Matlab, ode113, for all dynamical simulations in the paper.

For the yellow (light gray) segments of the curves in Figs. 2(c) and 3(a) which are connected to the inter-site ring-DB (top of thick curve in Fig. 2(c), and (v) in Fig. 3(a)), the perturbed dynamics also shows tendencies for localization on a single site. It is however less stable for these segments, in the sense that certain perturbations indeed lead to localization, while others only result in large amplitude oscillations on all sites. By conducting a similar study as the one presented in Fig. 5(d) for the subfamily displayed in Fig. 2(c), it is revealed that for $\omega \gtrsim 0.55$ almost all simulations result in localization on either of the initially most occupied ring-sites. As $\omega$ is further decreased, the probability that localization does not occur increases, and for e.g. $\omega \approx 0.45$ approximately half of the simulations do not result in any localization.

In the parts of the curves with complex unstable eigenvalues, resulting in an oscillatory
FIG. 6: (Color online) Dynamics for the initial conditions \( \{ \psi_n(0), \psi_C(0) \} = \{ \delta_{n,k}, 0 \} \), for \( N_R = 6, \gamma = 1, \beta = 0.1155 \) and \( \delta = (a) 0.05, (b) 0.10, (c) 0.15 \). The black curve is the initially mostly occupied ring-site, the orange (dark gray) curve is the central site, and the yellow (light gray) curve is the summation of \( |\psi_n|^2 \) over all other ring-sites. As \( \delta \) approaches the value where the DBs become linearly unstable, the dynamics stops being trapped on the initial ring-site.

instability, we have seen no tendency for the perturbed dynamics to become localized, but there are instead rather large amplitude oscillations on all sites.

C. Dynamics of approximate Discrete Breathers

The linear stability of the exact DB solutions also gives an indication of when some simpler, and perhaps experimentally more realistic, initial conditions remain localized. Fig. 6 displays the dynamics for the initial condition \( \{ \psi_n(0), \psi_C(0) \} = \{ \delta_{n,k}, 0 \} \), corresponding to sending in light in only one ring-site, for \( N_R = 6, \gamma = 1, \beta = 0.1155 \) and different values of \( \delta \). Localization on the initially occupied ring-site survives up to values of \( \delta \) comparable to when the DB-family becomes linearly unstable, i.e. the point marked with (i) in Figs. 2(a) and 3(a). This result holds also for other choices of localized initial conditions, e.g. Gaussian and exponentially localized. It is also necessary that \( \beta \) is not too big for \( \{ \psi_n(0), \psi_C(0) \} = \{ \delta_{n,k}, 0 \} \) to remain localized in one site, because otherwise the corresponding exact DB is too spread out over the ring-sites. For \( N_R = 6 \) and \( \gamma = 1 \), as in Fig. 6, localization disappears at \( \beta \approx 0.16 \) for \( \delta = 0.10 \) and \( \beta \approx 0.19 \) for \( \delta = 0.05 \).

In order for the initial condition \( \{ \psi_n(0), \psi_C(0) \} = \{ \delta_{n,k}/\sqrt{2}, \pm 1/\sqrt{2} \} \) to remain localized on two sites, \( \beta \) and \( \delta \) must be quite small (of the order of \( \sim 0.01 \)), and \( \delta \) and \( \psi_C \) must have
the opposite sign. This is both because the corresponding DB is more spread out for a given \( \beta \) compared to the one which is empty on the central site, and because the DB-families only are stable in one direction close to the solutions marked with (ii) in Figs. 2 and 3 for very small \( \delta \). Ref. [19] studies the dynamics of this initial condition.

D. The split-bifurcation

Let us now turn our attention to the nature of the split-bifurcation. Fig. 7(a)-(c) shows the DB-subfamilies in the \( \delta - \omega \) plane for \( N_R = 6, \gamma = 1 \) and \( \beta = 0.1386 \) [Fig. 7(a)] and \( \beta = 0.1485 \) [Fig. 7(b)-(c)], located on either side of the split-bifurcation (at \( \beta \approx 0.148 \)). The subfamilies of dimerlike solutions, which these DB-subfamilies bifurcate with above the split-bifurcation, are included as yellow (light gray) curves in Fig. 7(a)-(c) (there are more than one family of dimerlike solutions, characterized by different ratios between the ring and central site [15, 16]). The two subfamilies of the on-site and inter-site ring-DBs bifurcate with a dimerlike solution, at the same value of \( \delta \) marked with black squares in Fig. 7(b)-(c), above the split-bifurcation. This is analogous to the \( \delta = 0 \) case (for \( N_R \geq 6 \)), where the on-site and inter-site ring-DBs bifurcate with the uniform-ring solution at the same value of \( \beta \) [26]. The lower black and orange (dark gray) curves in Fig. 7(b)-(c) bifurcate with a dimerlike solution in the same family, but for a different \( \delta \) marked with black circles. These curves are the subfamilies of the on-site and inter-site DBs with nonzero central amplitude, respectively.

From Fig. 7(a)-(c) we conclude that the split-bifurcation occurs when the two subfamilies of the on-site and inter-site ring-DBs, being separate subfamilies below the split-bifurcation for even \( N_R \), merge together at a single point (and its mirror image), associated with a dimerlike solution. This point is located at \( \delta \approx 0.05 \) for the parameter-values in Fig. 7. The point splits up in two as \( \beta \) is further increased, so that the subfamilies of the on-site and inter-site ring-DBs bifurcate with the dimerlike solution at some \( |\delta| \) (squares in Fig. 7(b)-(c)), while the other subfamilies (bottom black and orange (dark gray) curves in Fig. 7(b)) bifurcate at another (circles in Fig. 7(b)-(c)).

The split-bifurcation scenario is similar for an odd number of ring-sites. The main difference is that the two points which merge together at the split-bifurcation belong to the same subfamily below the split-bifurcation [cf. Fig. 3(a)], since the on-site and inter-site ring-DB
FIG. 7: (Color online) $N_R = 6$ and $\gamma = 1$. (a) DB-subfamilies for $\beta = 0.1386 < \beta_{\text{split}}^{}(N_R, \gamma)$, the black and orange (dark gray) lines are the subfamilies of the on-site and inter-site ring-DBs, respectively [cf. Fig. 2(a) and 2(c)]. The yellow (light gray) line is the subfamily of the dimerlike solution which the DBs bifurcate with above the split-bifurcation. (b) Similar plot for $\beta = 0.1485 > \beta_{\text{split}}^{}(N_R, \gamma)$, where the black and orange (dark gray) curves have split up. Bifurcations of the DBs from dimerlike solutions are marked with black circles and squares. (c) Zooms in on (b). (d) Absolute values of the stability eigenvalues for the subfamily of dimerlike solutions in (c), imaginary eigenvalues marked yellow (gray), and real (and zero) eigenvalues marked black. Two (degenerate) pairs of stability eigenvalues collide at the origin when the DBs bifurcate from the dimerlike solutions in (b) and (c), marked with vertical lines in (c) and (d).

are connected below the split-bifurcation for odd $N_R$.

Signatures of the split-bifurcation can also be seen when studying the stability eigenvalues of the dimerlike solution. The bifurcations of the DB-subfamilies with dimerlike solutions are associated with a collision between two (degenerate) pairs of stability eigenvalues at
FIG. 8: (Color online) Values of $\beta^{\text{split}}(N_R, \gamma)$ (black), $\beta^{\text{OS}}_{\text{close}}(N_R, \gamma)$ (orange or dark gray), and $\beta^{\text{IS}}_{\text{close}}(N_R, \gamma)$ (yellow or light gray) as functions of (a) $\gamma$ (for $N_R = 6$) and (b) $N_R$ (for $\gamma = 1$). $\beta^{\text{split}}(N_R, \gamma) \to \beta_{\text{ring}}(N_R)$ as $\gamma \to \infty$, meaning that $\beta^{\text{split}}(N_R, \gamma) \to 1/3$ in (a). The orange (gray) curve in (a) approaches $\beta \approx 0.27$ as $\gamma$ becomes large.

the origin of the complex plane, illustrated in Fig. 7(d). The collision is between two pairs since two separate subfamilies bifurcate from the dimerlike solution. There are thus six eigenvalues equal to zero at the bifurcation points, two present for all solutions and associated with a global phase rotation (because of the U(1) symmetry of the model), while the additional four are connected to the bifurcation. The colliding pairs of eigenvalues are imaginary in the region between the two bifurcations ($0.0411 < \delta < 0.0536$ for Fig. 7(d)) and real otherwise. The interval between the two bifurcations shrinks towards zero when decreasing $\beta$, so that at $\beta = \beta^{\text{split}}(N_R, \gamma)$ the curves $\lambda(\delta)$ for these two pairs become tangent to $\lambda = 0$ at $\delta \approx 0.05$. It should also be noted that the family of dimerlike solutions which the DB-subfamilies bifurcate with is linearly unstable. Note that there exists another family of dimerlike solutions which is linearly stable, characterized by a larger central amplitude [16].

Fig. 8 shows how the value of $\beta^{\text{split}}(N_R, \gamma)$, $\beta^{\text{OS}}_{\text{close}}(N_R, \gamma)$ and $\beta^{\text{IS}}_{\text{close}}(N_R, \gamma)$ change as functions of $\gamma$ (for $N_R = 6$), and $N_R$ (for $\gamma = 1$). For sufficiently small $\gamma$ ($\gamma \lesssim 0.18$ in Fig. 8(a)) $\beta^{\text{split}}(N_R, \gamma) = 0$, and all subfamilies of the on-site and inter-site ring-DBs
bifurcate with dimerlike solutions. In the opposite limit $\gamma \to \infty$, $\beta_{\text{split}}(N_R, \gamma) \to \beta_{\text{ring}}(N_R) = \lim_{\gamma \to \infty} \beta_{\text{dimer}}(N_R, \gamma)$ [Eq. (7)], so that no subfamily bifurcates with a dimerlike solution. This can be understood since $\phi_C \to 0$ at $\delta = 0$ when $\gamma \to \infty$ according to (6a), and so DBs with and without central amplitudes approach each other, leading to a vanishing of the upper loop in figures such as Figs. 2(a) and 3(a), where the split occurs. The orange (gray) curve for $\beta_{\text{close}}^O(N_R, \gamma)$ in Fig. 8(a) approaches a value $\beta_{\text{close}}^O(N_R, \gamma) \approx 0.27$ as $\gamma$ becomes large (determined numerically). For $N_R \leq 6$ no close-bifurcation occurs for the subfamily of the inter-site ring-DB for any $\gamma$, so that the yellow (light gray) curve in Fig. 8(a) is constant $= 0$. Note that $\beta_{\text{split}}(N_R, \gamma)$ has a maximum value at $N_R = 13$ and 14 in Fig. 8(b), and that the interval between the split-bifurcation and the close-bifurcations, and thus the region where the DB-subfamilies form closed loops, decreases as $N_R$ increases.

VI. SUMMARY AND CONCLUSIONS

We have demonstrated the existence of Discrete Breathers (DBs) for a DNLS model with the geometry of a ring coupled to a single central site, a model which may describe e.g. an optical multicore fiber. The DBs under consideration are localized on a few sites of the ring, and possibly also on the central site.

By numerically following families of DBs from the anti-continuous limit, we have been able to determine under which parameter conditions these solutions exist. Given values of the ring to central site coupling $\delta$, the central-site nonlinearity $\gamma$, and the number of sites, a ‘split-bifurcation’ occurs for a certain value of the coupling between neighboring ring-sites $\beta = \beta_{\text{split}}(N_R, \gamma)$. For $\beta > \beta_{\text{split}}(N_R, \gamma)$ the DB-subfamilies bifurcate with a dimerlike solution, where all ring-sites have the same amplitude, while for $\beta < \beta_{\text{split}}(N_R, \gamma)$ they form closed loops in the $\delta - \omega$ parameter space ($\omega$ being the DBs’ frequency), connecting different DBs at $\delta = 0$ with zero and nonzero central amplitudes, respectively. Other ‘close-bifurcations’ occur for $\beta < \beta_{\text{split}}(N_R, \gamma)$. Below these close-bifurcations, the DB-subfamilies do not form closed loops, but instead converge towards certain solutions for large $|\delta|$. For instance, the subfamily of the on-site ring-DB converges towards a single-peak DB with all amplitudes shifted down. The topology of the loops existing between the close- and split-bifurcations depends on whether there is an even or odd number of ring-sites. The subfamilies typically bifurcate with the dimerlike solution for large values of $\beta$. 

22
with the exception of fewer than 6 ring-sites, where another close-bifurcation occurs and the subfamilies return to forming closed loops in the \( \delta - \omega \) parameter space for sufficiently large values of \( \beta \) (however still converging to the dimerlike solution for intermediate \( \beta \)). We have also shown how the DBs change along different parts of the subfamily curves.

The linear stability of the DBs has also been studied. The DBs will remain linearly stable when followed from the on-site ring-DB at \( \delta = 0 \) (corresponding to a single excited ring-site in the anti-continuous limit) to a value of \( |\delta| \) where the subfamilies make a bend in the \( \delta - \omega \) plane. These stable solutions will have the largest part of the norm located in the ring. The DBs close to the on-site symmetric DB with central occupation (corresponding to exciting the central site and one ring-site in the anti-continuous limit) will also be linearly stable, but only for rather small values of \( |\delta| \), and under the appropriate phase relation between central site and ring, given by the sign of \( \delta \).

The dynamics of some simpler initial conditions, which approximate the exact DBs, has also been examined. When starting with one excited ring-site and all others empty, \( \beta \) cannot be too big if the norm should remain mainly localized on the initial site, because otherwise the corresponding exact DB solution gets too spread out. Also, \( \delta \) must be smaller than a value comparable to when the DBs become unstable.

We think that our results would be realizable with multicore fibers, and possibly also in other experimental settings such as BECs in a ‘painted’ optical potential [33] or optically induced photonic lattices, and hope that this work will inspire further studies and experimental confirmations.

Acknowledgments

The authors would like to thank Aleksandra Maluckov for attracting our interest to this particular geometry of the DNLS model.


