Seminar 6: COUPLED HARMONIC OSCILLATORS

1. Lagrangian and Equations of Motion

Let consider a system consisting of two harmonic oscillators that are coupled together. As a model, we will use two particles attached to elastic strings, as shown in Figure from the Set of Problems. For simplicity, we assume that the oscillators are identical and are restricted to move in a straight line. The coupling between oscillators is represented by a spring of stiffness K'. The system has two degrees of freedom, and we thus need to use two generalized coordinates to represent the configuration of the system. The natural choice is x_1 and x_2, the displacement of the particles from their equilibrium positions.

The Lagrangian of the system is

$$L = T - V = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 - \frac{1}{2}Kx_1^2 - \frac{1}{2}K'(x_2 - x_1)^2 - \frac{1}{2}Kx_2^2. \tag{1}$$

To derive Lagrange's equations of motion we calculate

$$\begin{cases} \frac{\partial L}{\partial \dot{x}_1} = m\dot{x}_1, \\ \frac{\partial L}{\partial \dot{x}_2} = m\dot{x}_2, \\ \frac{\partial L}{\partial x_1} = -Kx_1 + K'(x_2 - x_1) = -(K + K')x_1 + K'x_2, \\ \frac{\partial L}{\partial x_2} = -Kx_2 - K'(x_2 - x_1) = -(K + K')x_2 + K'x_1, \end{cases} \tag{2}$$

which yields

$$\begin{cases} \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1} - \frac{\partial L}{\partial x_1} = m\ddot{x}_1 + (K + K')x_1 - K'x_2; \\ \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_2} - \frac{\partial L}{\partial x_2} = m\ddot{x}_2 + (K + K')x_2 - K'x_1. \end{cases} \tag{3}$$

Therefore, Lagrange's equations of motion are as follows:

$$\begin{cases} m\ddot{x}_1 + (K + K')x_1 - K'x_2 = 0; \\ m\ddot{x}_2 - K'x_1 + (K + K')x_2 = 0, \end{cases} \tag{4}$$

or

$$\begin{cases} T_{11}\ddot{x}_1 + T_{12}\ddot{x}_2 + V_{11}x_1 + V_{12}x_2 = 0; \\ T_{21}\ddot{x}_1 + T_{22}\ddot{x}_2 + V_{21}x_1 + V_{22}x_2 = 0. \end{cases} \tag{5}$$

where we introduce the matrices

$$T \equiv \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \tag{6}$$

and

$$V \equiv \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} = \begin{pmatrix} K + K' & -K' \\ -K' & K + K' \end{pmatrix}. \tag{7}$$

Note that Eqs. (5) can be rewritten in a form of a single matrix equation

$$T\ddot{\vec{\eta}} + V\vec{\eta} = 0, \tag{8}$$

where $\vec{\eta}$ is the column matrix (or vector) whose components represents the configuration or the *state* of the system as a whole,

$$\vec{\eta} \equiv \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{9}$$

The matrix equation (8) in an explicit component form is

$$\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \begin{pmatrix} K+K' & -K' \\ -K' & K+K' \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0. \tag{10}$$

Finally, using Einstein convention we can write this equation also in the form

$$T_{ij}\ddot{\eta}_j + V_{ij}\eta_j = 0 \quad (i, j = 1, 2). \tag{11}$$

Note that in any form these equations are coupled: say, in the initial form (5) the cross terms are nonzero, or in matrix form (8) the $\mathbf{V}$-matrix has nonzero off-diagonal elements.

2. Solution of Lagrange's equations

Let search for the solution of Eq. (8) in the form

$$\vec{\eta} = \mathbf{a}\cos(\omega t - \delta), \tag{12}$$

whose components are therefore

$$x_1 = a_1 \cos(\omega t - \delta), \quad x_2 = a_2 \cos(\omega t - \delta). \tag{13}$$

Hence we search for the solution whose components has both the same frequency and phase but a different amplitude. Equation (8) with the assumed solution (13) becomes

$$V\mathbf{a} = \omega^2 T\mathbf{a}, \tag{14}$$

or

$$\begin{pmatrix} K+K' & -K' \\ -K' & K+K' \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \omega^2 \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}. \tag{15}$$

We are well familiar with equations of such type [*see:* Eq. (3.19) in Lecture Notes]. This is nothing but the *eigenvalue equation* with respect to the *eigenvector* $\mathbf{a}$, and ω^2 is its *eigenvalue.* This equation is equivalent to a system of linear, homogeneous equations given by

$$\begin{pmatrix} K+K'-\omega^2 m & -K' \\ -K' & K+K'-\omega^2 m \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0. \tag{16}$$

These equations have a nontrivial solution (that is, solutions other than $a_1 = a_2 = 0$) if and *only* if

$$\begin{vmatrix} K + K' - \omega^2 m & -K' \\ -K' & K + K' - \omega^2 m \end{vmatrix} = 0. \tag{17}$$

On expanding this determinant, we obtain

$$(K + K' - \omega^2 m)^2 - K'^2 = 0. \tag{18}$$

Rearranging this as follows:

$$(K + K' - \omega^2 m)^2 - K'^2 = (\omega^2 m - K - K')^2 - K'^2$$
$$= (\omega^2 m - K - K' + K')(\omega^2 m - K - K' - K') = (\omega^2 m - K)[\omega^2 m - (K + 2K')] = 0, \tag{19}$$

we see that the roots of Eq. (17) are given by

$$\omega_1^2 = \frac{K}{m}, \quad \omega_2^2 = \frac{K + 2K'}{m}. \tag{20}$$

Next we may substitute the eigenfrequencies ω_1 and ω_2 back into equation (16) to find the relations between the components of the eigenvectors]. For convenience, we denote a specific eigenvector $\mathbf{a}_i$ $(i = 1, 2)$, so that the ith component of the jth eigenvector will be a_{ij} $(i, j = 1, 2)$ [Don't confuse a specific eigenvector $\mathbf{a}_i$ with the scalar component a_i of some generalized eigenvector]. For the first eigenvector, the matrix equation (16) becomes

$$\begin{pmatrix} K + K' - \omega_1^2 m & -K' \\ -K' & K + K' - \omega_1^2 m \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} = 0. \tag{21}$$

The first component of this matrix equation

$$[(K + K') - \omega_1^2 m]a_{11} - K' a_{21} = 0 \tag{22}$$

yields the solution

$$a_{11} = a_{21}. \tag{23}$$

For the second eigenvector, Eq. (22) changes to

$$[(K + K') - \omega_2^2 m]a_{12} - K' a_{22} = 0, \tag{24}$$

which yields components

$$a_{12} = -a_{22}. \tag{25}$$

Substituting these eigenvectors into Eq. (12), we obtain two particular solutions

$$\begin{cases} \vec{\eta}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} a_{11} \cos(\omega_1 t - \delta_1), \\ \vec{\eta}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} a_{22} \cos(\omega_2 t - \delta_2), \end{cases} \tag{26}$$

whose sum determines the general solution

$$\vec{\eta} = \vec{\eta}_1 + \vec{\eta}_2 = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \tag{27}$$

where

$$\begin{cases} x_1(t) = A_1 \cos(\omega_1 t - \delta_1) - A_2 \cos(\omega_2 t - \delta_2) \\ x_2(t) = A_1 \cos(\omega_1 t - \delta_1) + A_2 \cos(\omega_2 t - \delta_2). \end{cases} \tag{28}$$

Here we introduced two new constants A_1 and A_2, such that $A_1/A_2 = a_{11}/a_{12}$. The constants A_1, A_2, δ_1 and δ_2 must be considered as the four unknows which might be determined from the initial conditions for the positions and velocities of each mass. To apply these conditions we may differentiate the functions $x_1(t)$ and $x_2(t)$ to find the velocities

$$\begin{cases} \dot{x}_1(t) = -\omega_1 A_1 \sin(\omega_1 t - \delta_1) + A_2 \omega_2 \sin(\omega_2 t - \delta_2) \\ \dot{x}_2(t) = -\omega_1 A_1 \sin(\omega_1 t - \delta_1) - \omega_2 A_2 \sin(\omega_2 t - \delta_2). \end{cases} \tag{29}$$

3. Initial Conditions and Results

Consider a specific initial configuration when the first particle is held at $x_1 = 0$, while the second particle is displaced one unit to the right, and then they both are released from rest. To describe this specific configuration we first notice that at time $t = 0$ Eqs. (28) and (29) become

$$\begin{cases} x_1(0) = A_1 \cos\delta_1 - A_2 \cos\delta_2 \\ x_2(0) = A_1 \cos\delta_1 + A_2 \cos\delta_2. \end{cases} \tag{30}$$

and

$$\begin{cases} \dot{x}_1(0) = \omega_1 A_1 \sin\delta_1 - A_2 \omega_2 \sin\delta_2 \\ \dot{x}_2(0) = \omega_1 A_1 \sin\delta_1 + \omega_2 A_2 \sin\delta_2. \end{cases} \tag{31}$$

Solving these equation with respect to the amplitudes and phase shifts, we obtain

$$\begin{cases} A_1^2 = \frac{1}{4}[x_1(0) + x_2(0)]^2 + \frac{1}{4\omega_1^2}[\dot{x}_1(0) + \dot{x}_2(0)]^2 \\ A_2^2 = \frac{1}{4}[x_2(0) - x_1(0)]^2 + \frac{1}{4\omega_2^2}[\dot{x}_2(0) - \dot{x}_1(0)]^2 \end{cases} \tag{32}$$

and

$$\begin{cases} \tan\delta_1 = \frac{\dot{x}_1(0) + \dot{x}_2(0)}{\omega_1[x_1(0) + x_2(0)]} \\ \tan\delta_2 = \frac{\dot{x}_2(0) - \dot{x}_1(0)}{\omega_2[x_2(0) - x_1(0)]} \end{cases}. \tag{33}$$

In a specific conditions of our problems

$$x_1(0) = 0, \quad x_2(0) = 1, \quad \dot{x}_1(0) = \dot{x}_2(0) = 0, \tag{34}$$

and the general relations (32) and (33) are reduced to

$$A_1 = A_2 = \frac{1}{2}, \quad \delta_1 = \delta_2 = 0. \tag{35}$$

Inserting these into Eqs. (28) yields

$$\begin{cases} x_1(t) = \frac{1}{2}(\cos\omega_1 t - \cos\omega_2 t) \\ x_2(t) = \frac{1}{2}(\cos\omega_1 t + \cos\omega_2 t) \end{cases}. \tag{36}$$

Note that the signs of A_1 and A_2 are taken in such a way that Eqs. (34) are satisfied.

The expressions (36) describe the well-known phenomenon of beats because x_1 and x_2 are represented by the sum and difference of two simple, equal-amplitude, harmonic motions whose frequencies are different. Moreover, these sum and difference can be easily transformed to the products of sine and cosine of a sum and a difference of frequencies, namely

$$x_1(t) = \sin\left[\frac{1}{2}(\omega_1 + \omega_2)t\right] \sin\left[\frac{1}{2}(\omega_1 - \omega_2)t\right] \tag{37}$$

and

$$x_2(t) = \cos\left[\frac{1}{2}(\omega_1 + \omega_2)t\right] \cos\left[\frac{1}{2}(\omega_1 - \omega_2)t\right]. \tag{38}$$

Figure 11.3.2 (see hard copy which will be given during the seminar) shows the functions x_1 and x_2 for the spring constants $K = 4$ and $K' = 1$ (in arbitrary units. The motion has been plotted over one complete period. We see that the amplitude of oscillations of the first mass slowly builds up and then dies away in step with the dying away and buildup of the amplitude of oscillations of the second mass. That is exactly which we call the *beats.*

Next it is instructive to define new time-dependent variables according to:

$$\begin{cases} \zeta_1 = \frac{1}{\sqrt{2}}\cos\omega_1 t \\ \zeta_2 = \frac{1}{\sqrt{2}}\cos\omega_2 t. \end{cases} \tag{39}$$

In terms of these variables

$$\begin{cases} x_1 = \frac{1}{\sqrt{2}}(\zeta_1 - \zeta_2) \\ x_2 = \frac{1}{\sqrt{2}}(\zeta_1 + \zeta_2), \end{cases} \tag{40}$$

or, in matrix notation,

$$\vec{\eta} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} \equiv A\vec{\zeta}. \tag{41}$$

By this, we introduce the matrix A,

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}. \tag{42}$$

This relation can be easily inverted to obtain ζ_1 and ζ_2 as functions of x_1 and x_2,

$$\vec{\zeta} = \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \equiv A^{-1}\vec{\eta}. \tag{43}$$

By this, we define the inverse matrix A^{-1},

$$A^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}. \tag{44}$$

Indeed you can easily check that

$$A^{-1}A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I, \tag{45}$$

as it might be.

Note that the matrix A and A^{-1} describe $\pm\pi/4$ rotations of a two dimensional coordinate system. This suggest that we can interpret x_1 and x_2 or ζ_1 and ζ_2 as components of one and the same vector $\mathbf{q}$ whose eindpoint represents the time-dependent configuration of the coupled oscillator in either of two different ccordinate systems, as shown in Figure 11.3.3 (in hard copy). As times goes on, the eindpoint of $\mathbf{q}$ traces out a path in either of two configuration spaces, (x_1, x_2) or (ζ_1, ζ_2). These path's are shown in Figure 11.3.4 (in hard copy), (a) and (b). We see that in both cases the trajectory of a system point is confined to a box. But in (x_1, x_2)-space the boundaries of the box make 45° lines with the coordinate axes, while in ζ_1, ζ_2-space they are parallel to the axes. This suggests that ζ_1, ζ_2-space is better choice to express the equations of motion of the system. But even more fascinating thing here is that, although the coordinates x_1 and x_2 exchange by their energies, ζ_1 and ζ_2 not, because they are functions only single frequences ω_1 and ω_2. These peculiarities is inherent in the *normal coordinates*, and we can thus call (ζ_1, ζ_2) the *normal coordinates* for the system at hand.

4. Normal Modes

Up to now, we dealt with the specific problem when the boundary conditions were given by Eq. (34). In process, we introduced the normal coordinates ζ_1, ζ_2 according to (39). As a result, we obtained the relations between initial coordinates x_1, x_2 and normal coordinates given by Eqs. (40). Now we suppose that these equations are valid for arbitrary initial conditions, and consider two particular cases

$$x_1(0) = x_2(0) = 1, \quad \dot{x}_1(0) = \dot{x}_2(0) = 0, \tag{46}$$

and

$$x_2(0) = -x_1(0) = 1, \quad \dot{x}_1(0) = \dot{x}_2(0) = 0. \tag{47}$$

In both cases the two masses are initially displaced from their equilibrium positions by equal amounts but in the same direction in the case of Eq. (1.46) and in opposite directions in the case of Eq. (47).

Let consider both cases in more detail. Substituting Eq. (46) into Eq. (40), we obtain

$$\begin{cases} \zeta_1(0) = \sqrt{2} \\ \zeta_2(0) = 0, \end{cases} \tag{48}$$

that implies the time-dependent solution

$$\begin{cases} \zeta_1(t) = \sqrt{2}\cos\omega_1 t \\ \zeta_2(t) = 0. \end{cases} \tag{49}$$

Then Eq. (40) yields

$$x_1(t) = x_2(t) = \cos\omega_1 t. \tag{50}$$

That is the two masses vibrate back and forth as though they were completely independent simple harmonic oscillators with identical frequencies, $\omega_1 = \sqrt{K/M}$. The system is said to execute a *normal mode* of oscillations called the *symmetric mode* which is pictured in Figure 11.3.5 (in hard copy).

On the other hand, in the case of the boundary conditions given by Eq. (47) we obtain

$$\begin{cases} \zeta_1(0) = 0 \\ \zeta_2(0) = \sqrt{2}, \end{cases} \tag{51}$$

that implies the time-dependent solution

$$\begin{cases} \zeta_1(t) = 0 \\ \zeta_2(t) = \sqrt{2}\cos\omega_2 t \end{cases} \tag{52}$$

Then Eq. (40) yields

$$x_2(t) = -x_1(t) = \cos\omega_2 t. \tag{53}$$

In this case each mass vibrates 180° out of phase with the other at the single frequency $\omega_2 = \sqrt{(K + 2K')/m}$, as shown in Figure 11.3.6 (in hard copy). Naturally, this normal mode of oscillations is called the *antisymmetric*, or, for obvious reason, the *breathing* mode. Notice that in the vibration of this kind no energy can be transferred from one mass to the other across the central point which is called a *nodal point* in the vibration.

So we see that the system of the two coupled harmonic oscillators can be started off such that the two masses vibrate at a single fixed frequency and never exchange energy. There are two ways for this but in both cases one of the normal coordinate is chosen to be zero while the other oscillates with one of the eigenfrequencies, ω_1

or ω_2. Such a situation implies the existence of a two separate differential equations for the normal coordinates ζ_1 and ζ_2 whose solutions represent two decoupled simple harmonic oscillators. We can guess that these equations might be obtained directly from Lagrange's equations by transforming the Lagrangian to a function of the normal coordinates.

5. Transformation to Normal Coordinates

To find such a transformation, we first rewrite the kinetic and potential energies given by Eq. (1) in a slightly different form introducing the row vectors

$$\tilde{\vec{\eta}} = (x_1, x_2), \quad \tilde{\dot{\vec{\eta}}} = (\dot{x}_1, \dot{x}_2). \tag{54}$$

Then the kinetic and potential energies become

$$\begin{aligned} T &= \tfrac{1}{2}m\dot{x}_1^2 + \tfrac{1}{2}m\dot{x}_2^2 = \\ &= \tfrac{1}{2}(\dot{x}_1, \dot{x}_2)\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \tfrac{1}{2}\tilde{\dot{\vec{\eta}}}T\dot{\vec{\eta}} \end{aligned} \tag{55}$$

and

$$\begin{aligned} V &= \tfrac{1}{2}(K+K')x_1^2 + \tfrac{1}{2}(K+K')x_2^2 - K'x_1x_2 \\ &= \tfrac{1}{2}(x_1, x_2)\begin{pmatrix} K+K' & -K' \\ -K' & K+K' \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \tfrac{1}{2}\tilde{\vec{\eta}}V\vec{\eta}. \end{aligned} \tag{56}$$

Applying the transformation between the vectors $\vec{\eta}$ and $\vec{\zeta}$ given by Eq. (41), we obtain

$$\begin{aligned} T &= \tfrac{1}{2}\widetilde{A\dot{\vec{\zeta}}}TA\dot{\vec{\zeta}} = \tfrac{1}{2}\tilde{\dot{\vec{\zeta}}}\tilde{A}TA\dot{\vec{\zeta}} = \\ &= \tfrac{1}{4}(\dot{\zeta}_1, \dot{\zeta}_2)\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \dot{\zeta}_1 \\ \dot{\zeta}_2 \end{pmatrix} \\ &= \tfrac{1}{2}(\dot{\zeta}_1, \dot{\zeta}_2)\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\begin{pmatrix} \dot{\zeta}_1 \\ \dot{\zeta}_2 \end{pmatrix} \\ &= \tfrac{1}{2}m\dot{\zeta}_1^2 + \tfrac{1}{2}m\dot{\zeta}_2^2 \end{aligned} \tag{57}$$

and

$$\begin{aligned} V &= \tfrac{1}{2}\widetilde{A\vec{\zeta}}VA\vec{\zeta} = \tfrac{1}{2}\tilde{\vec{\zeta}}\tilde{A}VA\vec{\zeta} = \\ &= \tfrac{1}{4}(\zeta_1, \zeta_2)\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} K+K' & -K' \\ -K' & K+K' \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} \\ &= \tfrac{1}{2}(\zeta_1, \zeta_2)\begin{pmatrix} K & 0 \\ 0 & K+2K' \end{pmatrix}\begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} \\ &= \tfrac{1}{2}K\zeta_1^2 + \tfrac{1}{2}(K+2K')\zeta_2^2. \end{aligned} \tag{58}$$

Here the use of the matrix identity $\widetilde{A\vec{\zeta}} = \tilde{\vec{\zeta}}\tilde{A}$ has been made.

Hence the Lagrangian is

$$L = T - V = \frac{1}{2}m\dot{\zeta}_1^2 + \frac{1}{2}m\dot{\zeta}_2^2 - \frac{1}{2}K\zeta_1^2 - \frac{1}{2}(K + 2K')\zeta_2^2. \tag{59}$$

As anticipated, it contains no cross terms, and the resulting equations of motion are

$$\begin{cases} m\ddot{\zeta}_1 + K\zeta_1 = 0, \\ m\ddot{\zeta}_2 + (K + 2K')\zeta_2 = 0, \end{cases} \tag{60}$$

or in matrix notation

$$\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \begin{pmatrix} \ddot{\zeta}_1 \\ \ddot{\zeta}_2 \end{pmatrix} + \begin{pmatrix} K & 0 \\ 0 & K + 2K' \end{pmatrix} \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} = 0. \tag{61}$$

These equations describe the motion of two uncoupled, simple harmonic oscillators, and their solutions are

$$\begin{cases} \zeta_1 = B_1 \cos(\omega_1 t - \delta_1), \\ \zeta_2 = B_2 \cos(\omega_2 t - \delta_2), \end{cases} \tag{62}$$

where

$$\omega_1^2 = \frac{K}{m}, \quad \omega_2^2 = \frac{K + 2K'}{m}. \tag{63}$$

Substituting the solutions (62) into the transformation formula (41), we obtain

$$\begin{cases} x_1 = \eta_1 = \frac{1}{\sqrt{2}}(\zeta_1 - \zeta_2) = A_1 \cos(\omega_1 t - \delta_1) - A_2 \cos(\omega_2 t - \delta_2) \\ x_2 = \eta_2 = \frac{1}{\sqrt{2}}(\zeta_1 + \zeta_2) = A_1 \cos(\omega_1 t - \delta_1) + A_2 \cos(\omega_2 t - \delta_2). \end{cases} \tag{64}$$

These expressions are identical with those of Eq. (28), obtained by direct solving the starting coupled equations written in coordinates η_1, η_2 (the factor $1/\sqrt{2}$ factor has been absorbed).

6. Diagonalization of Lagrangian

Note that both T and V matrices in the normal coordinate representation (see the matrix equation (61)) are diagonal, and that each of those matrices was diagonalized by the so-called *congruent transformation*

$$T_{diag} = \tilde{A}TA \quad V_{diag} = \tilde{A}TA, \tag{65}$$

where the matrix A has been defined by Eq. (42). But notice: in view of the expressions given by Eqs. (41) and (43)) this matrix is equivalent to the matrix whose *its columns are the* (x_1, x_2 *components of the eigenvectors* $\mathbf{a}_i$, that is,

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}. \tag{66}$$

Each $\mathbf{a}_i$ in this expression is a column vector which obeys the equation

$$V\mathbf{a}_i = \omega_i^2 T\mathbf{a}_i \quad (i = 1, 2), \tag{67}$$

where a_{ij} refers to the jth component of the eigenvector $\mathbf{a}_i$, and ω_i^2 is its eigenfrequency.

Therefore, the matrix formed with each eigenvector as one of its column is the desired matrix A that transforms the generalized coordinates into normal coordinates and diagonalizes the V and T matrices that make up the Lagrangian of the system. Moreover, the elements of the diagonal matrices T_{diag} and V_{diag} determine the eigenfrequencies by means of the relations

$$\omega_i^2 = \frac{\tilde{\mathbf{a}}_i V \mathbf{a}_i}{\tilde{\mathbf{a}}_i T \mathbf{a}_i} \quad (i = 1, 2) \tag{68}$$

which follow directly from Eq. (67).

Everything sounds well, isn't it? However, pay your attention, please, in what way we became so clever as were able to find the matrix which diagonalizes the Lagrangian. *First*, we solved the coupled equations of motion in their original form, and *second* we find the normal mode representation of these solution. In other words, we were forced to solve the coupled equations of motion *before* finding the normal modes that we need to decouple them. The question arises: Can we obtain the normal modes first in a way, other than by "educated guess-work"? Unfortunately, there is no general prescription how to do this but one method works quite well in many cases of interest. The method is based on the use of some symmetry properties of the Lagrangian for finding the normal coordinates.

To see how it works, let take the Lagrangian for any two-component coupled oscillator in the following general form:

$$L = \frac{1}{2}T_{11}\dot{x}_1^2 + \frac{1}{2}T_{22}\dot{x}_2^2 + T_{12}\dot{x}_1\dot{x}_2 - \frac{1}{2}V_{11}x_1^2 - \frac{1}{2}V_{22}x_2^2 - V_{12}x_1x_2. \tag{69}$$

Now suppose, in this Lagrangian we replace x_2 with $\pm\alpha x_1$ and x_1 with $\pm x_2/\alpha$, and after this operation the Lagrangian remains the same. Then it is said to be *invariant under an exchange operation.* Carrying out the exchange

$$\alpha x_1 \to x_2, \quad x_2/\alpha \to x_1 \tag{70}$$

transforms the Lagrangian (69) to

$$L' = \frac{1}{2}T_{11}\frac{\dot{x}_2^2}{\alpha^2} + \frac{1}{2}T_{22}\alpha^2\dot{x}_1^2 + T_{12}\dot{x}_2\dot{x}_1 - \frac{1}{2}V_{11}\frac{x_2^2}{\alpha^2} - \frac{1}{2}V_{22}\alpha^2 x_1^2 - V_{12}x_2x_1. \tag{71}$$

We see that the two cross terms in L' are identical to those in L, and hence the invariance property will held if

$$\alpha^2 = \frac{T_{11}}{T_{22}} = \frac{V_{11}}{V_{22}}. \tag{72}$$

The second of these equalities,

$$\frac{T_{11}}{T_{22}} = \frac{V_{11}}{V_{22}}, \tag{73}$$

must be a property of the Lagrangian for the system under consideration, while the first condition,

$$\alpha^2 = \frac{T_{11}}{T_{22}}, \tag{74}$$

determines the ratio of the x-components that may be used for the Lagrangian will be invariant under exchange operation. Namely, x_2-component must be $\pm\alpha$ times that of the first.

This suggests the two eigenvectors $\mathbf{a_1}$ and $\mathbf{a_2}$ of the form

$$\mathbf{a_1} = \begin{pmatrix} 1 \\ \alpha \end{pmatrix} \quad \mathbf{a_2} = \begin{pmatrix} -1 \\ \alpha \end{pmatrix}, \tag{75}$$

and thus the A matrix that generates the transformation from generalized to normal coordinates has to be taken as

$$A = (\mathbf{a_1}\,\mathbf{a_2}) = \begin{pmatrix} 1 & -1 \\ \alpha & \alpha \end{pmatrix}. \tag{76}$$

It is instructive to reexamine the problem of a two coupled oscillator in light of this discussion. Looking at the diagonal elements of the matrices T and V in Eqs. (6) and (7), we see that the condition (73) is satisfied automatically and the condition (74) requires $\alpha = 1$. Hence

$$A = (\mathbf{a_1}\,\mathbf{a_2}) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \tag{77}$$

and the matrices T and V are diagonalized according to the transformations

$$\begin{aligned} T_{diag} &= \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \\ &= 2 \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \end{aligned} \tag{78}$$

and

$$\begin{aligned} V_{diag} &= \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} K+K' & -K' \\ -K' & K+K' \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \\ &= 2 \begin{pmatrix} K & 0 \\ 0 & K+2K' \end{pmatrix}. \end{aligned} \tag{79}$$

The ratio of the diagonal elements of V_{diag} and T_{diag} yield the eigenfrequences ω_1^2 and ω_2^2 obtained previously [*see: Eqs. (62)*]. Note that the multiplicative factor 2 that occurs in Eqs. (78) and (79) cancels out in these ratios and is therefore irrelevant. If we wish, it could be eliminated by normalizing the eigenvectors $\mathbf{a_1}$ and $\mathbf{a_2}$ by the factor $1/\sqrt{2}$ (in fact, we did this early).