METHOD OF IMAGES

Under favorable conditions it is possible to infer from the geometry of the situation that a small number of suitable placed charges of appropriate magnitudes, external to the region of interest, can simulate the required boundary conditions.

Examples are:
one or more charges on one side of a planar boundary,
one or more charges outside a spherical boundary,
one or more charges inside a spherical boundary,

Point charge in the presence of a grounded conducting sphere.

We have \( q \) outside the sphere, a distance \( y \) from the origin. We will show that it is enough to place one image charge \( q' \) inside the sphere a distance \( y' \) from the origin to fulfil the boundary condition \( \Phi = 0 \) at the spherical surface. From symmetry considerations this image charge must lie on the line connecting the origin and the position of \( q \).
The potential outside the spherical surface from the two charges is:

$$\Phi(x) = \frac{q}{|x-y|} + \frac{q'}{|x-y'|}$$

$$= \frac{q}{|x n - y n'|} + \frac{q'}{|x n' - y' n'|}$$

On the boundary it is

$$\Phi(a) = \frac{q}{a |n - \frac{y}{a} n'|} + \frac{q'}{y' |n' - \frac{a}{y} n|}$$

We see that with the choice

$$q = -q', \quad y = \frac{a}{y'}, \quad a = \frac{a}{y}$$

makes the potential vanish on the boundary.

This means

$$q' = -\frac{a}{y} q, \quad y' = \frac{a^2}{y}$$

Since the solution to Laplace's equation outside the sphere fulfilling the boundary conditions is unique, we have found the potential outside the sphere:

$$\Phi(x) = \frac{q}{|x-y|} - \frac{a q}{y |x - \frac{a^2}{y^2} y|}; \quad x \geq a$$

We have replaced the conducting sphere with the induced charge density with a single image charge. The two charges $q$ and $q'$ produce the same potential for $x \geq a.$
Since we have the correct potential we may calculate the induced charge density on the sphere of the real problem:

$$\sigma(\theta, \varphi) = - \frac{1}{4\pi} \frac{\partial \Phi(x, \theta, \varphi)}{\partial x} \bigg|_{x=a}$$

According to Gauss' law the total induced charge must be equal to the image charge, and it is.

The induced force on the charge \(q\) from the sphere can be written down directly using Coulomb's law and the image charge:

$$F = \frac{qq'}{(y-y')^2} \frac{y}{y}$$

$$= -q \left( \frac{a}{y} \right) \frac{q^2}{(y-a^2/y)^2} \frac{y}{y}$$

$$= -q \left( \frac{a}{y} \right)^2 \left[ 1 - \left( \frac{a}{y} \right)^2 \right]^{-2} \frac{y}{y}$$

Alternatively the force can be found as minus the force from \(q\) acting on the surface of the sphere.

**Discussion**: Note that the force is attractive at all separations between the charge and the sphere. The induced charge density has opposite sign compared to \(q\) all over the sphere.
Point charge in the presence of a charged, insulated, conducting sphere.

\[ \Phi(x) = \frac{q}{|x-y|} - \frac{aq}{y|x-a^2/y^2|} + \frac{Q + \frac{a}{y}q}{|x|}; \quad x \geq a \]

The charge \(Q-q'\) will distribute itself uniformly over the sphere.

\[ F = \frac{q}{y^2} \left[ Q - \frac{q a^3 (2y^2 - a^2)}{y(y^2 - a^2)^2} \right] \frac{y}{y} \]

Discussion: Here the force can be repulsive at large separations if the net charge of the sphere, \(Q\), has the same sign as the charge \(q\). If \(Q = 0\) the induced charge density has the same sign as \(q\) on the backside of the sphere.

Point charge in the presence of a conducting sphere at fixed potential.

\[ \Phi(x) = \frac{q}{|x-y|} - \frac{aq}{y|x-a^2/y^2|} + \frac{Va}{|x|}; \quad x \geq a \]

\[ F = \frac{q}{y^2} \left[ Va - \frac{q a y^3}{(y^2 - a^2)^2} \right] \frac{y}{y} \]

Discussion: We have now obtained the induced force between the charge and the conducting sphere for different boundary conditions. If we want to calculate the energy of the system, the work needed to bring the charge from infinity to its position near the sphere, how should we do?
One way is to perform the calculation:

\[ W = -\int_{y}^{\infty} F(r)dr = \int_{y}^{\infty} F(r)dr \]

Can we put \( W(y) = q\Phi(y) \)?

The answer is no! First we have to neglect the first term in the expressions for the potential in our examples. This term represents the self interaction of the charge itself. The last term in the the last example is O.K. to use as it is. The other terms are induced and should be multiplied with 1/2.

An alternative way to treat the problem is to study the energy density in the field. Using the expression \( q \) times the potential corresponds to integrating the energy density over the whole space using the expression valid only outside the sphere. Thus we also include the contribution inside the sphere. This contribution is not real and should be omitted. This part contains half the energy. Just integrating outside the sphere gives the right contribution.

All what we have derived above is also valid for charge \( q \) inside the sphere. The only difference is that now the expression for the potential is valid only inside the sphere and the total induced surface charge density is \(-q\), not \( q'\). The expression for the surface charge density should be multiplied by -1 since the surface normal now points inwards.
Conducting sphere in a uniform electric field by method of images.

We study a conducting sphere of radius $a$ in a uniform electric field $E_0$.

We may view the uniform electric field as originating from two charges of opposite sign and on opposite side of the sphere at infinite separation. The electric field is then

$$E_0 = \frac{2Q}{R^2}$$

or

$$Q = \frac{E_0 R^2}{2}$$

where $R$ will go to infinity in the end.

$$\Phi(r, \theta) = -E_0 r \cos \theta + \frac{aQ/R}{\sqrt{(r \sin \theta)^2 + (r \cos \theta - a^2/R)^2}}$$

$$+ \frac{-aQ/R}{\sqrt{(r \sin \theta)^2 + (r \cos \theta + a^2/R)^2}}$$
Letting $R$ go to infinity:

\[
\Phi(r, \theta) \to -E_0 r \cos \theta + \frac{aE_0 R}{2} \frac{1}{\sqrt{r^2 + a^4/R^2 - 2a^2 r \cos \theta / R}} - \frac{aE_0 R}{2} \frac{1}{\sqrt{r^2 + a^4/R^2 + 2a^2 r \cos \theta / R}} \\
\to -E_0 r \cos \theta + \frac{aE_0 R}{2r} \left[ \left( 1 + a^2 \cos \theta / rR \right) - \left( 1 - a^2 \cos \theta / rR \right) \right] \\
\to -E_0 r \cos \theta + \frac{aE_0 R}{2r} 2a^2 \cos \theta / rR \\
= -E_0 r \cos \theta + \frac{a^3 E_0}{r^2} \cos \theta = -E_0 \left( r - \frac{a^3}{r^2} \right) \cos \theta
\]

Note that the image charges form a dipole of strength

\[
D = \frac{aQ}{R} \frac{2a^2}{R} = \frac{2a^3}{R^2} Q = \frac{2a^3}{R^2} \frac{E_0 R^2}{2} = a^3 E_0
\]

The polarizability of a metallic sphere is just $a^3$. We will see this later.

The surface charge density is

\[
\sigma = -\frac{1}{4\pi} \left. \frac{\partial \Phi}{\partial r} \right|_{r=a} = \frac{3}{4\pi} E_0 \cos \theta
\]
Method of inversion.

What we have discussed suggests that there is some sort of equivalence of solutions of potential problems under the reciprocal radius transformation,

\[ r \to r' = \frac{a^2}{r} \]

This transformation is called *inversion in a sphere*. The radius of the sphere is called the *radius of inversion* and the center of the sphere, *center of inversion*.

The following is true:

Let \( \Phi(r, \theta, \phi) \) be the potential due to a set of point charges \( q_i \) at the points \( (r_i, \theta_i, \phi_i) \). Then the potential

\[ \Phi'(r, \theta, \phi) = \frac{a}{r} \Phi \left( \frac{a^2}{r}, \theta, \phi \right) \]

is the potential due to charges,

\[ q_i' = \frac{a}{r_i} q_i \]

located at the points \( \left( \frac{a^2}{r_i}, \theta_i, \phi_i \right) \).

Discussion: If the original problem is one where there are conducting surfaces held at constant potentials, the inverted surfaces will not in general have fixed potentials. The only exception is if the potential is kept zero.

Another thing to be aware of is that there may appear point charges at the center of inversion after the inversion.

Spheres are spheres or planes after the inversion and planes are mapped into spheres that passes through the center of inversion.
Planar interfaces

The results found so far can easily be used to find the image charges for planar interfaces, just by letting the radius of the sphere go to infinity. Let the charge $q$ be placed to the right of the interface at a distance $x$. The image charge $q'$ will be placed to the left at $-x'$. This means that

$$y = a + x;$$
$$y' = a - x'$$

Put this into the relations for $q'$ and $y'$ and let $a$ go to infinity.

$$q' = -\frac{a}{y}q = -\frac{a}{a + x}q \rightarrow -q;$$
$$a - x' = y' = \frac{a^2}{y} = \frac{a^2}{a + x} \rightarrow \frac{a^2(1 - x/a)}{a} = a - x$$

Thus the image charge is $-q$ and it is placed at the same distance from the interface as $q$ but on the other side.
METHOD OF IMAGES AT THE BOUNDARY BETWEEN DIELECTRICS

In this case we no longer have the boundary condition that the potential is constant at the interface. We now must use the conditions that the normal components of the $D$-fields and parallel components of the $E$-fields are continuous. We furthermore have fields on both sides of the boundary.

Let us put the charge $q$ in $A$ at the distance $d$ from the interface and place an image charge in $A'$ at the same distance from the interface. The potential from these two charges on the right side is

$$
\Phi(\rho, z) = \frac{1}{\varepsilon_2} \left( \frac{q}{\sqrt{\rho^2 + (z - d)^2}} + \frac{q'}{\sqrt{\rho^2 + (z + d)^2}} \right), \quad z > 0
$$

To get the potential on the left side we put an effective charge $q''$ in $A$ to find the potential (replacing $q$)

$$
\Phi(\rho, z) = \frac{1}{\varepsilon_1} \frac{q''}{\sqrt{\rho^2 + (z - d)^2}}, \quad z < 0
$$
From this we find
\[
\frac{\partial \Phi}{\partial z} \bigg|_{z=0^+} = \frac{1}{\varepsilon_2} \frac{(q - q')d}{\left(\rho^2 + d^2\right)^{3/2}}
\]
\[
\frac{\partial \Phi}{\partial z} \bigg|_{z=0^-} = \frac{1}{\varepsilon_1} \frac{q'' d}{\left(\rho^2 + d^2\right)^{3/2}}
\]
\[
\frac{\partial \Phi}{\partial \rho} \bigg|_{z=0^+} = -\frac{1}{\varepsilon_2} \frac{(q + q')\rho}{\left(\rho^2 + d^2\right)^{3/2}}
\]
\[
\frac{\partial \Phi}{\partial \rho} \bigg|_{z=0^-} = -\frac{1}{\varepsilon_1} \frac{q'' \rho}{\left(\rho^2 + d^2\right)^{3/2}}
\]

The boundary conditions give
\[
q - q' = q'' ;
\]
\[
\frac{1}{\varepsilon_2} (q + q') = \frac{1}{\varepsilon_1} q''
\]
and
\[
q' = -\left(\frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2}\right) q ;
\]
\[
q'' = \left(\frac{2\varepsilon_1}{\varepsilon_1 + \varepsilon_2}\right) q
\]

Discussion: From this we could have obtained the results for a conducting plane if the metal were treated as a dielectric with infinite static dielectric function \( \varepsilon_1 = \infty \); \( q' = -q \) and \( q'' = 2q \). Note that \( q'' \) does not contribute to the potential inside the metal since the potential is divided by \( \varepsilon_1 \) which is infinite. I do not think we can handle the problem with a charge outside a dielectric sphere with a finite number of image charges.