REFLECTION FROM A METALLIC SURFACE

For a metallic medium the dielectric function and the index of refraction are complex valued functions. This is also the case for semiconductors and insulators in certain frequency ranges near and at absorption bands. Fresnel's equations are still valid but the angles in the equations are now complex valued and do no longer have the obvious geometrical interpretation.

For normal incidence we have

$$R = \left(\frac{n_2 - n_1}{n_2 + n_1} \right)^2 \rightarrow \left| \frac{\tilde{n}_2 - \tilde{n}_1}{\tilde{n}_2 + \tilde{n}_1} \right|^2 = \frac{(n_2 - n_1)^2 + (k_2 - k_1)^2}{(n_2 + n_1)^2 + (k_2 + k_1)^2}$$

where

$$\tilde{n} = n + ik = n(1 + i\kappa)$$

where κ is the so called extinction coefficient.

For metallic systems

$$\tilde{n} = \sqrt{\tilde{\varepsilon}} = \sqrt{\varepsilon + 4\pi i\sigma/\omega}$$

For normal incidence the refracted, or rather transmitted, wave will vary as

$$\mathbf{E}_2 \sim e^{i(\tilde{n}k_0 z - \omega t)} = e^{-kk_0 z} e^{i(nk_0 z - \omega t)}$$

REFRACTION INTO A CONDUCTING MEDIUM

Naively one would believe that the plane wave inside the metal would vary like in the preceding section but in a new direction. The damping would be along the direction of propagation.

This is however not so! The damping is still in the z-direction only. This means that the surfaces of constant amplitude are no longer the same as the surfaces of constant phase.

The surfaces of constant amplitude are parallel to the interface and the surfaces of constant phase are perpendicular to the direction of propagation.

The whole problem is a bit awkward. Approximately the angle of refraction and the wavelength are determined by the real part of the index of refraction and the damping in the z-direction by the imaginary part of the index of refraction. Note however that this is just approximate!

The correct dependence is

$$
\begin{aligned}
\mathbf{E}_2 &\sim e^{i\left(k_0 \sin\theta_0(-x)+k_0\sqrt{\tilde{n}^2-\sin^2\theta_0}\,z-\omega t\right)} \\
&= e^{-k_0 \operatorname{Im}\sqrt{\tilde{n}^2-\sin^2\theta_0}\,z}\, e^{i\left(k_0 \sin\theta_0(-x)+k_0 \operatorname{Re}\sqrt{\tilde{n}^2-\sin^2\theta_0}\,z-\omega t\right)}
\end{aligned}
$$

and the real transmission angle is

$$
\theta_2^{real} = \arctan\left[\frac{\sin\theta_0}{\operatorname{Re}\sqrt{\tilde{n}^2-\sin^2\theta_0}}\right]
$$

Note that Fresnel's coefficients are still valid for complex valued refractive indices but the angles are complex valued.

TOTAL INTERNAL REFLECTION

I will just briefly touch upon this subject, because of lack of time. It is very important though. I treat this in more detail in my other course, TFYY70 fundamentals of surface modes.

If an electromagnetic wave is impinging on an interface from a material of higher refractive index to a one with lower, the refraction angle is greater than the angle of incidence.

If we let the angle of incidence increase we reach a critical angle, $\theta_c \equiv \sin^{-1}\left(n_2/n_1\right)$, when all energy is reflected. For all angles greater than this critical angle we have total reflection. The interesting thing is that there is also a wave parallel to the interface, which decays exponentially away from the interface. If we have a steady state condition and an infinite interface all energy is reflected.

If we start the irradiation, at first some energy is used to create this evanescent wave but as soon as steady state conditions are reached no more energy is needed for this.

If however the interface is finite this evanescent wave can radiate out at the edge of the interface and some energy is needed to compensate for this.

There can also be imperfections at the interface making the wave radiate.

There can also be damping or losses in e.g. a metallic system which means that energy has to be fed into the mode.

There are other interesting effects arising from that the mode has fields extending outside the interface. If the interface is a glass-prism–air–interface the fields extend in the air outside the prism. Putting another prism close to the first allows the mode to "jump" across the air gap and be emitted through the second prism.

Similar effects are utilized in the so called ATR-experiment, used to study surface modes.

MULTILAYERS

Now we will study the reflection and transmission between two parallel interfaces between three media. It could for instance be an anti reflection coating on an air-glass interface. We have such a problem to solve in the problem solving session. I want to demonstrate that the problem can be solved in different ways and also show how the solution can be extended to the case of many layers.

The first approach is a straight forward extension of what we have done for the single interface.

In the first medium we have an incoming wave and a reflected. In the sandwiched layer we also have two waves, one going in the positive z-direction and one in the negative z-direction. In the third medium we only have one wave travelling in the positive z-direction.

We use the same type of boundary conditions as we used in the single interface geometry at both boundaries and find the amplitudes and angles for the waves.

The procedure is straight forward and can be extended to a geometry with more than two interfaces. However, the solution becomes increasingly cumbersome the more interfaces we have.

We will use this method in the problem solving session and will not dwell on it here. Instead we use another method. All the waves in our ansatz, except the incoming wave on the first interface, can be viewed as a superposition of waves having been multiple reflected a varying number of times between the interfaces. The amplitudes of these waves, taking the phases into account are added together and this leads to interference effects

We will here follow the outline in *Surface Modes in Physics*[1].

[1]Bo E. Sernelius, *Surface Modes in Physic*, Wiley-VCH, Berlin 2001.

Fresnel's coefficients give the relative amplitudes of the reflected and transmitted waves at an interface between two media. We have derived these. They are:

$$t_{ij}^{s} = \frac{2\sin\theta_j \cos\theta_i}{\sin\left(\theta_i + \theta_j\right)} = \frac{2n_i \cos\theta_i}{n_i \cos\theta_i + n_j \cos\theta_j}$$

$$r_{ij}^{s} = -\frac{\sin\left(\theta_i - \theta_j\right)}{\sin\left(\theta_i + \theta_j\right)} = \frac{n_i \cos\theta_i - n_j \cos\theta_j}{n_i \cos\theta_i + n_j \cos\theta_j}$$

$$t_{ij}^{p} = \frac{2\sin\theta_j \cos\theta_i}{\sin\left(\theta_i + \theta_j\right)\cos\left(\theta_i - \theta_j\right)} = \frac{2n_i \cos\theta_i}{n_j \cos\theta_i + n_i \cos\theta_j}$$

$$r_{ij}^{p} = \frac{\tan\left(\theta_i - \theta_j\right)}{\tan\left(\theta_i + \theta_j\right)} = \frac{n_j \cos\theta_i - n_i \cos\theta_j}{n_j \cos\theta_i + n_i \cos\theta_j}$$

where the superscripts *s* and *p* represent *s*-polarized and *p*-polarized waves, respectively and the angles θ_i and θ_j are the angle of incidence and angle of transmission, respectively. The optical properties of each material enter in the form of the refractive index, *n*.

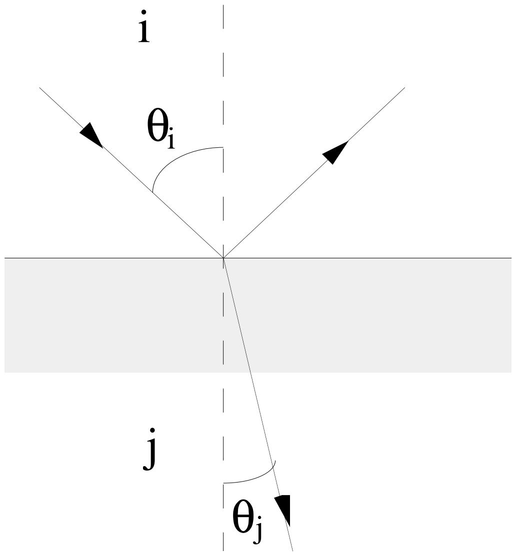
These coefficients are valid also for complex valued refractive indices. In that case the angles are not to be interpreted as the geometrical angles. They are complex valued, and the sine and cosine of the angles are also complex valued. The sine functions are obtained from Snell's law

$$n_j \sin\theta_j = n_i \sin\theta_i$$

and the cosine functions are obtained from relations of the type:

$$\cos\theta_i = \sqrt{1 - \sin^2\theta_i}$$

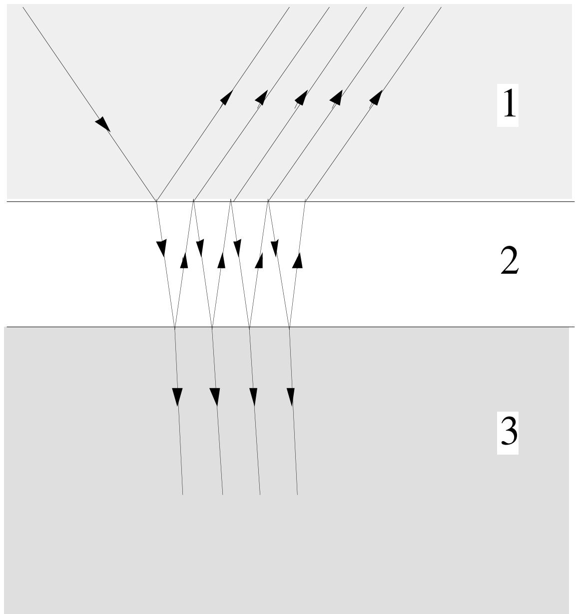
For *s*-polarized waves the electric field vector is perpendicular to the plane of incidence and for *p*-polarized waves it is in the plane of incidence. The plane of incidence is the plane defined by the incoming wave and the normal to the surface.



The interface between two media *i* and *j*, discussed in the text.

TWO INTERFACES BETWEEN THREE MEDIA

We treat two interfaces as shown in the figure below. We need to take multiple reflections in the middle layer into account.



Waves contributing to the total reflected and transmitted waves in a three layer structure as discussed in the text.

The total reflected amplitude, r, is obtained from an infinite summation of waves due to the multiple reflections in the middle layer.

Each time a wave impinges on an interface the Fresnel equations are used and the phases of the waves are taken into account. All coefficients are complex-valued.

The phase $2\delta_2$ is the phase difference for two waves: one wave that is transmitted through the first interface, passing through layer *2*, is reflected at the second interface, passing through layer *2* again and is finally transmitted through the first interface; the second wave is one that is reflected at the first interface. This second wave has furthermore traveled a longer distance in layer *1* before it impinges on the interface.

The results are obtained as follows:

$$r = r_{12} + t_{12} x t_{21};\ \ x = e^{i\delta_2} r_{23} e^{i\delta_2}\left[1 + r_{21} x\right];\ \ x = \frac{e^{i2\delta_2} r_{23}}{1 - e^{i2\delta_2} r_{23} r_{21}}$$

$$t = t_{12} y t_{23};\ \ y = \left[e^{i\delta_2} + y r_{23} e^{i\delta_2} r_{21} e^{i\delta_2}\right];\ \ y = \frac{e^{i\delta_2}}{1 - e^{i2\delta_2} r_{23} r_{21}}$$

$$\delta_2 = \frac{2\pi d_2}{\lambda} n_2 \cos(\theta_2)$$

$$r = r_{12} + \frac{e^{i2\delta_2} r_{23} t_{12} t_{21}}{1 - e^{i2\delta_2} r_{23} r_{21}} = \frac{r_{12} - r_{12} e^{i2\delta_2} r_{23} r_{21} + e^{i2\delta_2} r_{23} t_{12} t_{21}}{1 - e^{i2\delta_2} r_{23} r_{21}}$$

$$= \frac{r_{12} + e^{i2\delta_2} r_{23}\left(t_{12} t_{21} - r_{12} r_{21}\right)}{1 - e^{i2\delta_2} r_{23} r_{21}} = \frac{r_{12} + e^{i2\delta_2} r_{23}}{1 - e^{i2\delta_2} r_{23} r_{21}}$$

$$= \frac{r_{12} + e^{i2\delta_2} r_{23}}{1 + e^{i2\delta_2} r_{12} r_{23}}$$

$$t = \frac{e^{i\delta_2} t_{12} t_{23}}{1 + e^{i2\delta_2} r_{12} r_{23}}$$

We have made use of the following useful relations:

$$r_{21} = -r_{12}$$
$$t_{12} t_{21} - r_{12} r_{21} = 1$$

These results are valid for both polarization directions.
The reflection, transmission and absorption are

$$R = |r|^2;\ \ T = \begin{cases} |t|^2 \dfrac{\mathrm{Re}\left[(n_3)^* \cos\theta_3\right]}{\mathrm{Re}\left[(n_1)^* \cos\theta_1\right]}, & p-\text{polarization} \\ |t|^2 \dfrac{\mathrm{Re}(n_3 \cos\theta_3)}{\mathrm{Re}(n_1 \cos\theta_1)}, & s-\text{polarization} \end{cases} ;\ A = 1 - R - T$$

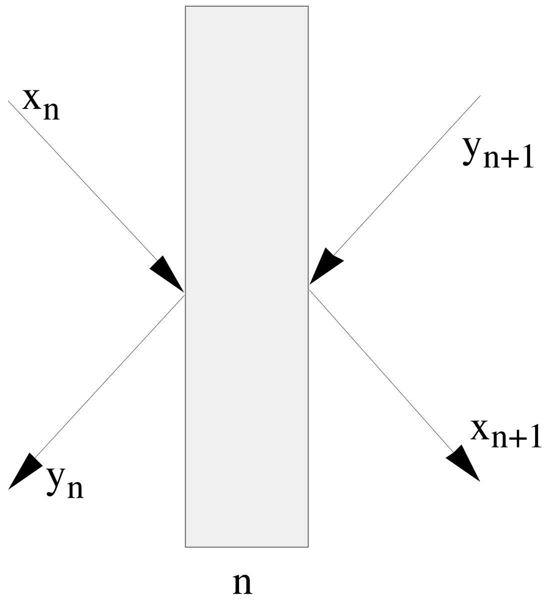
$$N^{eff} = \frac{\tilde{n}\sin\theta}{\sin\theta^{real}} = \frac{\sin\theta_0}{\sin\theta^{real}}\ ;$$

$$\sin\theta_0 = N^{eff} \sin\theta^{real}\ ;\ \ \mathrm{Re}\left[\tilde{\mathbf{k}}\cdot\hat{\mathbf{n}}\right] = \tfrac{\omega}{c} N^{eff} \cos\theta^{real}$$

GENERAL NUMBER OF LAYERS

The procedure described in the preceding section can be extended to more layers, but becomes too cumbersome if we have many layers. Here we will introduce a more suitable approach.

Assume that we have *N* layers sandwiched between medium 0 and *N*+1. So we have *N*+2 media. Then in general layer *n* will have an incoming and a reflected wave on the left side. We denote these with x_n and y_n, respectively. On the right hand side there will be an incoming and an outgoing wave as well. These are reflected and incoming waves on layer *n*+1.



The waves on the two sides of the layer are related to each other. This relation can be expressed as

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \tilde{\mathbf{M}}_n \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix}$$

where

$$\tilde{\mathbf{M}}_n = \frac{1}{t_{n-1,n}}\begin{pmatrix} 1 & r_{n-1,n} \\ r_{n-1,n} & 1 \end{pmatrix}\begin{pmatrix} e^{-i\delta_n} & 0 \\ 0 & e^{i\delta_n} \end{pmatrix}$$
$$= \frac{1}{t_{n-1,n}}\begin{pmatrix} e^{-i\delta_n} & r_{n-1,n}e^{i\delta_n} \\ r_{n-1,n}e^{-i\delta_n} & e^{i\delta_n} \end{pmatrix}$$

Each interface is shared between two neighboring layers. We have let the left most interface belong to the layer. This means that the fields to the right in the figure are the fields inside layer *n* just to the left of the right interface. To get the fields just inside layer *n*+1 we have to multiply from the right with the matrix

$$\frac{1}{t_{n,n+1}}\begin{pmatrix} 1 & r_{n,n+1} \\ r_{n,n+1} & 1 \end{pmatrix}$$

We have *N* number of layers and *N*+1 number of interfaces in our problem. This means that we in the end have to multiply with a matrix of this kind to take care of the right most interface.

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \tilde{\mathbf{M}}_1 \cdot \tilde{\mathbf{M}}_2 \cdot \ldots \tilde{\mathbf{M}}_n \cdot \ldots \tilde{\mathbf{M}}_N \cdot \frac{1}{t_{N,N+1}}\begin{pmatrix} 1 & r_{N,N+1} \\ r_{N,N+1} & 1 \end{pmatrix}\begin{pmatrix} x_{N+1} \\ y_{N+1} \end{pmatrix}$$
$$= \tilde{\mathbf{M}}\begin{pmatrix} x_{N+1} \\ y_{N+1} \end{pmatrix}$$

Thus if we know the fields on the right hand side of our layers we get the fields on the left hand side. This is not exactly what we want. We want y_1 and x_{N+1} as functions of x_1 when $y_{N+1} = 0$.

This is <u>not</u> impossible to find. We have:

$$x_1 = M_{11}x_{N+1} + M_{12}y_{N+1} = M_{11}x_{N+1}$$
$$y_1 = M_{21}x_{N+1} + M_{22}y_{N+1} = M_{21}x_{N+1}$$

and

$$t = \frac{x_{N+1}}{x_1} = \frac{1}{M_{11}}$$
$$r = \frac{y_1}{x_1} = \frac{M_{21}}{M_{11}}$$

Let us see if we reproduce our previous result with one layer. Then we have

$$\tilde{\mathbf{M}} = \frac{1}{t_{12}}\begin{pmatrix} e^{-i\delta_2} & r_{12}e^{i\delta_2} \\ r_{12}e^{-i\delta_2} & e^{i\delta_2} \end{pmatrix}\frac{1}{t_{23}}\begin{pmatrix} 1 & r_{23} \\ r_{23} & 1 \end{pmatrix}$$
$$= \frac{1}{t_{12}t_{23}}\begin{pmatrix} e^{-i\delta_2} + r_{12}r_{23}e^{i\delta_2} & r_{23}e^{-i\delta_2} + r_{12}e^{i\delta_2} \\ r_{12}e^{-i\delta_2} + r_{23}e^{i\delta_2} & r_{12}r_{23}e^{-i\delta_2} + e^{i\delta_2} \end{pmatrix}$$

and

$$M_{11} = \frac{e^{-i\delta_2} + r_{12}r_{23}e^{i\delta_2}}{t_{12}t_{23}}$$
$$M_{21} = \frac{r_{12}e^{-i\delta_2} + r_{23}e^{i\delta_2}}{t_{12}t_{23}}$$

and

$$t = \frac{t_{12}t_{23}}{e^{-i\delta_2} + r_{12}r_{23}e^{i\delta_2}} = \frac{e^{i\delta_2}t_{12}t_{23}}{1 + r_{12}r_{23}e^{2i\delta_2}}$$
$$r = \frac{r_{12}e^{-i\delta_2} + r_{23}e^{i\delta_2}}{e^{-i\delta_2} + r_{12}r_{23}e^{i\delta_2}} = \frac{r_{12} + r_{23}e^{i2\delta_2}}{1 + r_{12}r_{23}e^{i2\delta_2}}$$